Semantics of type theory and the simplicial model

Paolo Capriotti

June 15, 2024

Contents

1 Introduction

These notes derive from a course on the semantics of type theory that I have given at TU Darmstadt in 2020.

My objective with this course was to give a reasonably complete introduction to the so-called *standard model* of homotopy type theory in simplicial sets, which would be accessible to students with limited familiarity with category theory, and no prior knowledge of algebraic topologic or homotopical algebra.

The construction of the model of type theory in simplicial sets developed in these notes entirely bypasses the question of the existence of the corresponding model structure, and avoids the need to resort to geometric realisation for the definition of weak equivalences.

In fact, the notion of weak equivalence is avoided altogether, and replaced by that of equivalence of fibrations, which is more well-behaved and easier to define and to deal with type-theoretically.

The first model of type theory featuring a genuinely intensional equality type and (in current terminology) a univalent universe was the groupoid model of Hofmann and Streicher [12]. The idea was later improved to a model where types are families of ∞-groupoids, realised as Kan fibrations over simplicial sets [13], in a ZFC metatheory plus enough inaccessible cardinals.

Soon after, in relation to questions about the then conjectured *homotopy canonicity* property of homotopy type theory, the problem of developing a fully constructive model was raised. That lead to the development of the first model in cubical sets [3], followed by many others [6, 1].

Meanwhile, the question of whether the properties of the existing simplicial model could be proved constructively was settled negatively by [4]. Nevertheless, constructive alternatives to the simplicial model have since been developed.

The results in [9] imply the existence of a model with Σ, Π and equality types using an algebraic notion of Kan fibration. However, it is not known how to build a univalent universe in such a model.

On the other side of the spectrum, the models in [8, 10] are based on the standard notion of Kan fibration (and reduce to the standard model of [13] in the presence of excluded middle) but lack stability of Π-types.

I will not consider the issue of constructivity in these notes, but I will nevertheless make use of some of the innovations introduced in its pursuit, such as the notion of *strong homotopy equivalence* (definition 5.19) — taken from [9] — and the technique of using the *equivalence extension property* for proving fibrancy of the universe — introduced by [6] in the development of a cubical model, and later repurposed by [15] to more general models including simplicial ones.

Apart from repackaging existing constructions in a more elementary format, I make no originality claims in these notes. In order to follow the construction, a basic understanding of elementary category theory is required, but not much more than the definition of category, functor, natural transformation, and limit.

Presheaves play a fundamental role in these notes, for a couple of reasons. First, it is very fruitful to think of the structure of a model of type theory in terms of presheaves and operations between them. This is in essence a categorical interpretation of the idea of *logical framework*. See for example [2, 5, 16] for applications of these ideas.

Secondly, the main objective of these notes is to construct a model of homotopy type theory in simplicial sets, which are themselves a special kind of presheaves. Some (though not all) of the type-theoretic structures in simplicial sets are in fact instances of their more general counterparts in presheaves.

For those reasons, the first part of these notes (section 2) is dedicated to recalling some basic facts about presheaves, including a general version of the *nerve-realisation* adjunction, used multiple times in the following, and the equivalence between presheaves over a fixed presheaf X and presheaves on the category of elements of X, which is exploited throughout the rest of the notes.

In section 3, I will introduce the main notion of model of type theory which we will be using, namely that of *categories with families* (*cwf* s). Cwfs by themselves only model the basic rules of substitutions of types and terms, and therefore we need further structures to model the various type formers, mostly Σ, Π and equality types. These are introduced in section 4.

Finally, section 5 is dedicated to the main construction: that of the simplicial model of type theory. This is the most technically challenging section, since certain facts, such as the equivalence extension property or closure of Kan fibrations under Π-types, are essentially impossible to establish without adequate preparation in the combinatorics of anodyne extensions (cf. proposition 5.16).

I am indebted to the participants of the Budapest type theory seminar for their help and suggestions in the development of the program for this course. Special thanks go to Christian Sattler, who has helped me gain a better understanding of some of the ideas in these notes, and has suggested various improvements to the presentation. I am also grateful to the students of the course, in particular Amjad Saef, for reporting several mistakes and typos in earlier drafts.

2 Preliminaries

If $\mathcal{A}$ is a category, we will denote the category of presheaves on $\mathcal{A}$ by $\widehat{\mathcal{A}}$. The category $\widehat{\mathcal{A}}$ is complete and cocomplete, with limits and colimits computed pointwise. The Yoneda embedding $\mathcal{A} \to \widehat{\mathcal{A}}$ will be denoted by $\mathcal{A}[-]$. Given a presheaf X on $\mathcal{A}$, an element $x \in X(y)$, and a morphism $f\colon a \to b$ in $\mathcal{A}$, we will use the notation xf for $X(f)(x)$, assuming the functor X is clear. The postfix notation has the advantage that functoriality of X can be written simply as $(xf)g = x(f \circ g)$.

Let us recall the Yoneda lemma.

Lemma 2.1. *For all presheaves X on $\mathcal{A}$, and $a \in \mathcal{A}$, there is a natural bijection*

$$\widehat{\mathcal{A}}(\mathcal{A}[a], X) \cong X(a).$$

The Yoneda lemma says that the representable functor $\mathcal{A}[a]$ plays the role of a "walking element" of "shape" a. It is essentially a single abstract element of a generic functor, equipped with the minimal structure needed to make it into a presheaf. We get actual elements of presheaves by mapping out of the representable.

For example, consider the category of directed graphs, which is easily seen to be equivalent to the presheaf category on $\mathcal{A} = 0 \underset{t}{\overset{s}{\rightrightarrows}} 1$. Then the representable $\mathcal{A}[0]$ is a graph consisting of a single vertex, the "walking vertex", while $\mathcal{A}[1]$ is two vertices and an edge between them, the "walking edge". The Yoneda lemma says that vertices of a graph can be regarded as maps from the walking vertex, and similarly edges are maps from the walking edge.

In the following, we will assume that $\mathcal{A}$ is small.

If X is a presheaf on $\mathcal{A}$, we will denote by $\mathcal{A}/X$ the *category of elements* of X. A concise definition is that $\mathcal{A}/X$ is the full subcategory of $\widehat{\mathcal{A}}/X$ consisting of maps from a representable presheaf to X. The terminology is motivated by the Yoneda lemma, since elements of a functor are the same as maps from a representable. Explicitly, the objects of $\mathcal{A}/X$ are pairs (a, x), where a is an object of $\mathcal{A}$, and $x \in X(a)$ is an element of X. Morphisms $(a, x) \to (b, y)$ are given by morphisms $f \in \mathcal{A}(a, b)$ such that $x = yf$.

Lemma 2.2. *Given presheaves X and Y,*

$$\lim_{(a,x)\in\mathcal{A}/X} Y(a) \cong \widehat{\mathcal{A}}(X, Y).$$

This has a simple, but important, consequence.

Lemma 2.3. *Every presheaf is a* canonical *colimit of representables. Explicitly, if X is any presheaf on $\mathcal{A}$, we have:*

$$X \cong \operatorname*{colim}_{(a,x)\in\mathcal{A}/X} \mathcal{A}[a].$$

Proof. Let Y be an arbitrary presheaf. Then

$$\widehat{\mathcal{A}}(X, Y) \cong \lim_{(a,x)\in\mathcal{A}/X} Y(a) \cong \lim_{(a,x)\in\mathcal{A}/X} \widehat{\mathcal{A}}(\mathcal{A}[a], Y),$$

which shows that X satisfies the universal property of the required colimit. $\square$

In the graph example, this specialises to the statement that every graph can be obtained as a colimit of vertices and edges, glued according to the combinatorial data of the graph itself.

The category of presheaves $\widehat{\mathcal{A}}$ can be thought of as the *completion* of $\mathcal{A}$ under small colimits. We will now make this statement precise.

Let $i\colon \mathcal{A} \to \mathcal{E}$ be a functor to a category $\mathcal{E}$. Define a functor $N_i\colon \mathcal{E} \to \widehat{\mathcal{A}}$, called the *nerve functor* associated to i, as follows:

$$N_i(Y)(a) = \mathcal{E}(i(a), Y).$$

Lemma 2.4. *Let $\mathcal{E}$ be a cocomplete category, and $i\colon \mathcal{A} \to \mathcal{E}$ a functor. Then the functor N_i defined above has a left adjoint τ_i, called the* realisation functor *associated to i. The functor τ_i preserves colimits and $\tau_i(\mathcal{A}[a]) \cong i(a)$ and is uniquely determined up to natural isomorphisms by these two properties.*

Proof. Suppose first that we have such a functor τ_i. Then

$$\tau_i(X) \cong \tau_i \operatorname*{colim}_{(a,x)\in\mathcal{A}/X} \mathcal{A}[a] \cong \operatorname*{colim}_{(a,x)\in\mathcal{A}/X} \tau_i(\mathcal{A}[a]) \cong \operatorname*{colim}_{(a,x)\in\mathcal{A}/X} i(a),$$

so τ_i is uniquely determined up to natural isomorphism. To show existence, we can take this as a definition of τ_i. The following calculation shows that τ_i is left adjoint to N_i:

$$\begin{aligned}\mathcal{E}(\tau_i(X), Y) &= \mathcal{E}\Big(\operatorname*{colim}_{(a,x)\in\mathcal{A}/X} i(a), Y\Big)\\ &\cong \lim_{(a,x)\in\mathcal{A}/X} \mathcal{E}(i(a), Y)\\ &\cong \widehat{\mathcal{A}}(X, N_i(Y)).\end{aligned}$$

In particular, τ_i preserves colimits, and furthermore, for all objects $a \in \mathcal{A}$, $Y \in \mathcal{E}$, we have a natural isomorphism,

$$\mathcal{E}(\tau_i(\mathcal{A}[a]), Y) \cong \widehat{\mathcal{A}}(\mathcal{A}[a], N_i(Y)) \cong \mathcal{E}(i(a), Y),$$

and it follows from the Yoneda lemma that $\tau_i(\mathcal{A}[a]) \cong i(a)$. $\square$

Note that if i is fully faithful, then $N_i(i(a)) \cong \mathcal{A}[a]$, since

$$N_i(i(a))(a') \cong \mathcal{E}(i(a'), i(a)) \cong \mathcal{A}(a', a) = \mathcal{A}[a](a').$$

Lemma 2.5. *There is an equivalence of categories:*

$$\widehat{\mathcal{A}/X} \cong \widehat{\mathcal{A}}/X.$$

Proof. Let $i\colon \mathcal{A}/X \to \widehat{A}/X$ be the inclusion. Since $\widehat{A}/X$ is cocomplete, this determines a nerve-realisation adjunction. It remains to show that this adjunction is an equivalence. First of all, if $p\colon Y \to X$ is an object of $\widehat{A}/X$, we can write Y as a colimit of representables, and this determines an isomorphism

$$(Y, p) \cong \operatorname*{colim}_{(a,y)\in\mathcal{A}/Y} (\mathcal{A}[a], p_y),$$

where $p_y\colon \mathcal{A}[a] \to X$ is the component at y of the cocone corresponding to p. Furthermore, since i is fully faithful, $N_i(\mathcal{A}[a], x) = (\mathcal{A}/X)[a, x]$ for all $x \in X(a) \cong \mathcal{A}[a] \to X$.

Therefore, it is enough to show that N_i preserves colimits. If $(a, x) \in \mathcal{A}/X$ is fixed, then

$$N_i(Y, p)(a, x) \cong \{y \in Y(a) \mid p(y) = x\} \cong x^*(Y(a), p)$$

where $x^*\colon \mathsf{Set}/X(a) \to \mathsf{Set}$ is the functor that returns the fibre over x. Since p^* has a right adjoint, it preserves colimits. It follows that the functor $(Y, p) \mapsto N_i(Y, p)(a, x)$ preserves colimits, and therefore N_i does, as required. $\square$

3 Categories with families

Definition 3.1. A *category with families* (cwf) [7] is given by:

- A category $\mathcal{C}$ with a distinguished terminal object. Objects of $\mathcal{C}$ are called *contexts*.
- A presheaf Ty on $\mathcal{C}$. Elements of Ty are called *types*.
- A presheaf Tm over Ty. Elements of Tm are called *terms*.
- For all types $A \in \mathsf{Ty}(\Gamma)$, a representative of the functor $\mathcal{C}/\Gamma \to \mathsf{Set}$ given by $(\Delta, \sigma) \mapsto \mathsf{Tm}_\Delta(A\sigma)$.

The last point implicitly says that the functor is representable, and furthermore we are given a specified representative as part of the structure of a cwf. More explicitly, this means that for any type $A \in \mathsf{Ty}(\Gamma)$, there is a context $\Gamma.A$, called the *context extension* of Γ by A, and a morphism $p_A\colon \Gamma.A \to \Gamma$, satisfying the following property, which we will refer to as the *universal property of context extension*: to give a morphism $\Delta \to \Gamma.A$ is the same as to give

- a morphism $\sigma\colon \Delta \to \Gamma$;
- a term $t\colon \mathsf{Tm}_\Delta(A\sigma)$.

Morphisms of the form $p_A\colon \Gamma.A \to \Gamma$ for some type $A \in \mathsf{Ty}(\Gamma)$ are called *display maps*.

Note that context extension can be regarded as a functor $\mathsf{ext}\colon \mathcal{C}/\mathsf{Ty} \to \mathcal{C}^{[1]}$, where $\mathcal{C}^{[1]}$ is the category of arrows of $\mathcal{C}$. For readers familiar with the notion of Grothendieck fibration, it may be useful to observe that the universal property above implies that the functor ext is *Cartesian*, i.e. it preserves Cartesian morphisms.

When working with cwfs, we will often implictly identify a term $a \in \mathsf{Tm}_\Gamma(A)$ with the corresponding morphism $\Gamma \to \Gamma.A$ given by the definition of context extension. For example, if $B \in \mathsf{Ty}(\Gamma.A)$, we write $Ba \in \mathsf{Ty}(\Gamma)$ for the type obtained using functoriality of Ty on the morphism corresponding to the term a. We can think of the type Ba as the result of *substituting* a into B.

We will also liberally omit substitution along *weakening* morphisms, i.e. compositions of display maps. So, for example, if $A \in \mathsf{Ty}(\Gamma)$, we will simply write A instead of Ap_A for the corresponding type in $\mathsf{Ty}(\Gamma.A)$. Applying the universal property of context extension to the identity morphism $\Gamma.A \to \Gamma.A$ yields a term $v_A \in \mathsf{Tm}_{\Gamma.A}(A)$, which we will refer to as the *variable* of type A.

Lemma 3.2. *For any type $A \in \mathsf{Ty}(\Gamma)$, and morphism $\sigma \in \mathcal{C}(\Delta, \Gamma)$, there is a morphism $\sigma^+ \in \mathcal{C}(\Delta.A\sigma, \Gamma.A)$ that fits into a pullback square:*

$$\begin{tikzcd}
\Delta.A\sigma \arrow[r, "\sigma^+"] \arrow[d, "p_{A\sigma}"'] & \Gamma.A \arrow[d, "p_A"] \\
\Delta \arrow[r, "\sigma"'] & \Gamma.
\end{tikzcd}$$

Proof. Let Θ be an arbitrary context. Then

$$\begin{aligned}
\mathcal{C}(\Theta, \Delta.A\sigma) &\cong \coprod_{\tau \in \mathcal{C}(\Theta,\Delta)} \mathcal{C}/\Delta((\Theta, \tau), (\Delta.A\sigma, p_{A\sigma})) \\
&\cong \coprod_{\tau \in \mathcal{C}(\Theta,\Delta)} \mathsf{Tm}_\Theta(A\sigma\tau) \\
&\cong \coprod_{\tau \in \mathcal{C}(\Theta,\Delta)} \mathcal{C}/\Gamma((\Theta, \sigma\tau), (\Gamma.A, p_A)) \\
&\cong \{(\tau, \psi) \mid \tau \in \mathcal{C}(\Theta, \Delta), \psi \in \mathcal{C}(\Theta, \Gamma.A), p_A\psi = \sigma\tau\},
\end{aligned}$$

and the last set in the chain of isomorphisms is simply the set of cones for the above pullback diagram. Since naturality is clear, this shows that $\Delta.A\sigma$ satisfies the universal property of the pullback, and also defines the map σ^+ as a component of the limit cone. To conclude, it remains to show that $p_{A\sigma}$ is indeed the other component of the limit cone, which can be verified explicitly by tracing through the chain of isomorphisms starting with the identity $\Delta.A\sigma \to \Delta.A\sigma$. $\square$

3.1 Sets

The standard example is the category of sets. In the following, we will make liberal use of universes. We consider a universe $\mathcal{V}$ fixed, and call a set *$\mathcal{V}$-small* if it is contained in $\mathcal{V}$. We can now define a cwf structure on the category of sets as follows:

- the base category is Set, with an arbitrary choice of a one-element set as a terminal object;
- for a set Γ, types over Γ are families of $\mathcal{V}$-small sets indexed over Γ, i.e. functions $\Gamma \to \mathcal{V}$;
- for a set Γ, and a type $A \colon \Gamma \to \mathcal{V}$, terms of A are sections, i.e. families $(a_x)_{x \in \Gamma}$, where $a_x \in A(x)$.

Context extension is formed by taking pairs: $\Gamma.A$ is the set of pairs (x, a), where $x \in \Gamma$ and $a \in A(x)$. The universal property is immediate to verify.

3.2 Syntactic models

The definition of cwf is simply a more categorical formulation of the structural rules of type theory. For this reason, assuming a precisely defined syntax of type theory, one should get a corresponding *syntactic model* by taking contexts, types and terms to be their syntactic counterparts. Morphisms of contexts are given by tuples of terms. The category structure on context, as well as the presheaf structures on types and terms, are defined using substitution.

The verification that the above construction indeed forms a cwf depends on the details of the definition of the syntax, which can get pretty tedious. For this reason, we will not be attempting to make any of this precise. In some cases, one can get around these difficulties using a more semantic approach to syntax. Namely, one can define a category of cwf equipped with appropriate type-theoretic structure, and define the corresponding syntactic model by taking an initial object in this category.

In the following sections, we will be limiting ourselves to exploring how one can define this type-theoretic structure precisely, but we will avoid thinking about morphisms between "structured" cwfs, which means that technically we will not be able to speak of the syntactic model as an initial cwf with structure. Nevertheless, at least at an intuitive level, it is useful to keep syntactic models in mind, as they are bridges between the semantic and syntactic sides of type theory. For more details on how to define a syntactic model, [11] is a good reference.

3.3 Groupoids

Recall that a groupoid is a category where all the morphisms are invertible. As before, consider a universe $\mathcal{V}$ of sets fixed. Denote by Gpd the category of small groupoids (and functors), and by $\mathsf{Gpd}^{(\mathcal{V})}$ the full subcategory of groupoids whose set of arrows is $\mathcal{V}$-small (the set of objects is then automatically $\mathcal{V}$-small). Following the same idea as the cwf of sets, one can define a cwf of categories as follows:

- the base category is Gpd, the category of small groupoids;
- for a groupoid Γ, types over Γ are families of $\mathcal{V}$-small groupoids over Γ, i.e. functors $\Gamma \to \mathsf{Gpd}^{(\mathcal{V})}$.

We could now define terms directly and prove that with these definitions Gpd is a cwf. In fact, it is a useful exercise to do so, and the reader is invited to try it before reading the construction below. However, we take this opportunity to show a more general technique to get a correct definition of terms given contexts, types and context extension.

So far, we have contexts and types, so let us define context extension. Given a type $A\colon \Gamma \to \mathsf{Gpd}^{(\mathcal{V})}$, we can consider a generalisation of the category of elements of a presheaf that we have been using so far. If $f\colon \Gamma(x,y)$ is a morphism in Γ, let $f_*\colon A(x) \to A(y)$ be the functor given by functoriality of A. Note that there are two levels of functoriality at play here, which can get confusing: since A is a functor, it transforms morphisms in Γ into morphisms in $\mathsf{Gpd}^{(\mathcal{V})}$, and the latter are themselves functors between groupoids.

Now, define a category $\Gamma.A$ as follows:

- objects of $\Gamma.A$ are pairs (x,a), where x is an object of Γ, and a is an object of $A(x)$;
- a morphism $(x,a) \to (y,b)$ is given by a pair (f,g), where $f \in \Gamma(x,y)$, and $g \in A(y)(f_*(a),b)$.

Composition and identities are defined in the obvious ways. Note that if $A(x)$ happens to be a discrete groupoid for all x, then $\Gamma.A$ coincides with the category of elements of A, regarded as a presheaf on Γ (since Γ is a groupoid, contravariance does not matter). In fact, in this case the morphism g of a pair (f,g) is forced to be an identity, recovering the elementary definition of category of elements precisely.

There is an obvious functor $p_A\colon \Gamma.A \to \Gamma$, and it is not hard to verify that all this data assembles into a functor $\mathsf{ext}\colon \mathsf{Gpd}/\mathsf{Ty} \to \mathsf{Gpd}^{[1]}$. For Gpd to be a cwf such that ext is the context extension functor, we must have, in particular:

$$\mathsf{Tm}_\Gamma(A) \cong \mathsf{Gpd}/\Gamma((\Gamma,\mathsf{id}),\mathsf{ext}(\Gamma,A))$$

This suggests that we can take the above equation as the *definition* of Tm. Note that the right hand side is the set of *sections* of p_A, i.e. the set of functors $s\colon \Gamma \to \Gamma.A$ such that $p_A \circ s = \mathsf{id}$.

To prove that this definition of Tm possesses the correct universal property, we have to show, that for all groupoids Δ, and functors $\sigma\colon \Delta \to \Gamma$, there is a natural isomorphism

$$\mathsf{Gpd}/\Delta((\Delta,\mathsf{id}),(\Delta.A\sigma,p_{A\sigma})) \cong \mathsf{Gpd}/\Gamma((\Delta,\sigma),(\Gamma.A,p_A)).$$

This follows easily once we prove that lemma 3.2 holds in Gpd.

Lemma 3.3. *The following square of groupoids and functors*

$$\begin{tikzcd} \Delta.A\sigma \arrow[r] \arrow[d] & \Gamma.A \arrow[d] \\ \Delta \arrow[r] & \Gamma \end{tikzcd}$$

is a pullback.

Proof. This can be verified explicitly expanding the definition of context extension. □

It is also possible to extend this cwf structure to a full-blown model of type theory, in a sense that will be made precise later. This results in the groupoid model [12], which was the first example of a model of type theory with a genuinely intensional equality type structure (cf. section 4.3.2) and a univalent universe.

3.4 Presheaf models

Let $\mathcal{I}$ be a small category. We will equip the corresponding presheaf category $\widehat{\mathcal{I}}$ with a cwf structure. Again, let $\mathcal{V}$ be a universe of sets. For a presheaf Γ, a type over Γ is defined to be a presheaf on $\mathcal{I}/\Gamma$ whose values are all $\mathcal{V}$-small sets. The functorial action is as follows: given a morphism $\sigma \colon \Delta \to \Gamma$, and a type A over Γ, the type $A\sigma$ is given as the composition

$$\mathcal{I}/\Delta \to \mathcal{I}/\Gamma \xrightarrow{A} \mathcal{V},$$

where the first map is the functor that sends a pair (a, x) to $(a, \sigma(x))$.

Given a context Γ and type A, let $p_A \colon \Gamma.A \to \Gamma$ be the object of $\widehat{\mathcal{I}}/\Gamma$ corresponding to A via the equivalence of lemma 2.5. We can give a more explicit description of $\Gamma.A$:

Lemma 3.4. *For all objects $i \in \mathcal{I}$, we have*

$$(\Gamma.A)(i) \cong \coprod_{x \in \Gamma(i)} A(i, x),$$

and the morphism $\Gamma.A \to \Gamma$ corresponds to the first projection via the isomorphism.

Proof. It is enough to show the claim when A is a representable, in which case it follows immediately from the definitions. □

Just like the groupoid example, we have a definition of contexts, types and context extension, which means that the rest of the cwf structure on $\widehat{\mathcal{I}}$ follows once we prove that lemma 3.2 holds in this category.

Lemma 3.5. *Given a context morphism $\sigma\colon \Delta \to \Gamma$ and a type A on Γ, the following diagram of presheaves and natural transformations*

$$\begin{tikzcd} \Delta.A\sigma \arrow[r] \arrow[d] & \Gamma.A \arrow[d] \\ \Delta \arrow[r] & \Gamma \end{tikzcd}$$

is a pullback.

Proof. This can be proved via an explicit calculation using lemma 3.4, which we leave to the reader. Here we give a more abstract proof using the uniqueness part of lemma 2.4.

Consider the functor $\widehat{\mathcal{I}/\Gamma} \to \widehat{\mathcal{I}/\Delta}$ given by precomposition with σ, so that by definition $A\sigma$ is the image of A through this functor. It is easy to show directly that this functor preserves colimits. Denote by σ^* the functor $\widehat{\mathcal{I}}/\Gamma \to \widehat{\mathcal{I}}/\Delta$ given by pullback along σ. Since σ^* has a right adjoint, it also preserves colimits.

Therefore, it is again enough to show the claim when A is representable, where the verification is immediate. □

4 Type formers

The structure of a cwf can only model the so-called *structural* rules of type theory, i.e. those regarding substitution of types and terms. All the other rules have to be added as extra structure on a cwf. The approach taken in these notes is standard, and a similar presentation can be found in [11].

4.1 Σ-types

Let us begin with Σ-types. We can take the rules for Σ-types in the syntax and translate them directly into an algebraic structure on a cwf, by giving the following *preliminary* definition.

Definition 4.1. An *unstable Σ-type structure* on a cwf $\mathcal{C}$ is given by:

- for all contexts $\Gamma \in \mathcal{C}$ and types $A \in \mathsf{Ty}(\Gamma)$, $B \in \mathsf{Ty}(\Gamma.A)$, a type $\Sigma_\Gamma(A, B) \in \mathsf{Ty}(\Gamma)$;
- an isomorphism

$$\mathsf{Tm}_\Gamma(\Sigma_\Gamma(A, B)) \cong \coprod_{a \in \mathsf{Tm}_\Gamma(A)} \mathsf{Tm}_\Gamma(Ba).$$

The isomorphism of definition 4.1 can be understood in terms of the syntactic definition of Σ-types in terms of pairing and projections. Going left to right in the isomorphism corresponds to taking the two projections, while going the other way corresponds to forming a pair given two terms of the correct types. The fact that these two maps are inverses to each other correspond to the β and η rules.

The reason for the qualifier *unstable* is that such a structure is not well-behaved enough to model the Σ-types of the syntax. What is missing is something describing the behaviour of Σ-types over varying contexts.

For example, if $\sigma \in \mathcal{C}(\Delta, \Gamma)$ is any context morphism, and A, B types as in definition 4.1, there is a priori no relation between the types $\Sigma(A, B)\sigma$ and $\Sigma(A\sigma, B\sigma^+)$. However, syntactically there is no difference between the two, so the semantics should reflect that. Similar considerations apply to the rest of the structure.

To organise these stability requirements in a way that makes it easy to state them and reason about them, we are going to restructure definition 4.1 into a more principled formulation. First, define a presheaf $\mathsf{Ty}^{(2)}$ on $\mathcal{C}$ as follows:

$$\mathsf{Ty}^{(2)}(\Gamma) = \coprod_{A \in \mathsf{Ty}(\Gamma)} \mathsf{Ty}(\Gamma.A).$$

An element of $\mathsf{Ty}^{(2)}(\Gamma)$ is a pair of types A, B, with B depending on A. The action of a morphism σ of $\mathcal{C}$ on a pair (A, B) is given by $(A, B)\sigma = (A\sigma, B\sigma^+)$.

Correspondingly, we have a presheaf $\mathsf{Tm}^{(2)}$ on $\mathcal{C}/\mathsf{Ty}^{(2)}$ defined by:

$$\mathsf{Tm}^{(2)}_\Gamma(A, B) = \coprod_{a \in \mathsf{Tm}_\Gamma(A)} \mathsf{Tm}_\Gamma(Ba),$$

so that an element of $\mathsf{Tm}^{(2)}_\Gamma(A, B)$ is a pair of terms a, b of types A and Ba respectively. By applying the universal property of context extension twice, we get a natural isomorphism

$$\mathsf{Tm}^{(2)}_\Delta(A\sigma, B\sigma^+) \cong \mathcal{C}/\Gamma((\Delta, \sigma), (\Gamma.A.B, p_A p_B)).$$

Definition 4.1 can now be read in terms of $\mathsf{Ty}^{(2)}$, and it consists of

- a family of maps $\Sigma_\Gamma \colon \mathsf{Ty}^{(2)}(\Gamma) \to \mathsf{Ty}(\Gamma)$, indexed over all contexts Γ;
- a family of isomorphisms $\mathsf{Tm}_\Gamma(\Sigma(A, B)) \cong \mathsf{Tm}^{(2)}_\Gamma(A, B)$, indexed over Γ, A, B.

The above formulation makes it clear that the definition is missing naturality. We can therefore strengthen it in the obvious way and obtain a definition of stable Σ-types.

Definition 4.2. A (stable) Σ-*type structure* on a cwf $\mathcal{C}$ is given by:

- a natural transformation $\Sigma \colon \mathsf{Ty}^{(2)} \to \mathsf{Ty}$;

- a natural isomorphism $\mathsf{Tm} \circ \Sigma \cong \mathsf{Tm}^{(2)}$.

Note that in the above definition we have abused notation, and denoted again with Σ the functor $\mathcal{C}/\mathsf{Ty}^{(2)} \to \mathcal{C}/\mathsf{Ty}$ induced by the natural transformation Σ.

We can give a number of equivalent characterisations of a Σ-type structure. First, let us recall the notion of orthogonality of morphisms in a category.

Definition 4.3. Let $f\colon A \to B$ and $g\colon X \to Y$ morphisms in a category $\mathcal{A}$. We say that f is *left orthogonal* to g, or equivalently that g is *right orthogonal* to f, if for all commutative squares

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "v"'] & Y, \end{tikzcd}$$

there exists a unique *diagonal lift*, i.e. a morphism $\ell\colon B \to X$ that makes both triangles commute:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "v"'] \arrow[ur, "\ell"] & Y. \end{tikzcd}$$

Proposition 4.4. *Let $\Sigma\colon \mathsf{Ty}^{(2)} \to \mathsf{Ty}$ be a natural transformation, and*

$$i\colon \Gamma.A.B \to \Gamma.\Sigma(A,B)$$

a morphism over Γ, natural in $(\Gamma, A, B) \in \mathcal{C}/\mathsf{Ty}^{(2)}$. The following are equivalent:

(i) *i is an isomorphism;*

(ii) *the map $\mathsf{Tm}^{(2)}_\Gamma(A,B) \to \mathsf{Tm}_\Gamma(\Sigma(A,B))$ induced by i via the universal property of context extension is an isomorphism;*

(iii) *for all $(\Delta, \sigma) \in \mathcal{C}/\Gamma$, composition with i induces an isomorphism*

$$\mathcal{C}/\Gamma(\Gamma.\Sigma(A,B), \Delta) \cong \mathcal{C}/\Gamma.A(\Gamma.A.B, \Delta.A\sigma);$$

(iv) *for all types $X \in \mathsf{Ty}(\Gamma.\Sigma(A,B))$, the map*

$$i^*\colon \mathsf{Tm}_{\Gamma.\Sigma(A,B)}(X) \to \mathsf{Tm}_{\Gamma.A.B}(Xi)$$

is an isomorphism;

(v) *i is left orthogonal to all display maps.*

Proof. $(i) \leftrightarrow (ii)$ Since i is a morphism over Γ, i is an isomorphism if and only if, for all $(\Delta, \sigma) \in \mathcal{C}/\Gamma$, it induces an isomorphism

$$\mathcal{C}/\Gamma(\Delta, \Gamma.A.B) \cong \mathcal{C}/\Gamma(\Delta, \Gamma.\Sigma(A, B)).$$

Now, the latter condition clearly implies (ii) using the universal property of context extension. Conversely, assume (ii) holds. Then in particular

$$\mathsf{Tm}^{(2)}_{\Delta}(A\sigma, B\sigma^{+}) \cong \mathsf{Tm}_{\Delta}(\Sigma(A\sigma, B\sigma^{+})),$$

hence, by naturality of Σ,

$$\mathsf{Tm}^{(2)}_{\Delta}(A\sigma, B\sigma^{+}) \cong \mathsf{Tm}_{\Delta}(\Sigma(A, B)\sigma),$$

and applying the universal property of context extension yields the desired isomorphism.

$(i) \leftrightarrow (iii)$ By Yoneda, i is an isomorphism if and only if it induces an isomorphism

$$\mathcal{C}/\Gamma(\Gamma.\Sigma(A, B), \Delta) \cong \mathcal{C}/\Gamma(\Gamma.A.B, \Delta),$$

and the right hand side is isomorphic to $\mathcal{C}/\Gamma.A(\Gamma.A.B, \Delta.A\sigma)$ by lemma 3.2.

$(i) \to (iv)$ Obvious.

$(iv) \to (v)$ Let $p_X : \Delta.X \to \Delta$ be a display map, and suppose given a commutative square

$$\begin{tikzcd}
\Gamma.A.B \arrow[r, "u"] \arrow[d, "i"'] & \Delta.X \arrow[d, "p_X"] \\
\Gamma.\Sigma(A, B) \arrow[r, "v"'] & \Delta.
\end{tikzcd}$$

By lemma 3.2, we can factor this square as follows:

$$\begin{tikzcd}
\Gamma.A.B \arrow[r, "u'"] \arrow[d, "i"'] & \Gamma.\Sigma(A, B).Xv \arrow[r, "v^{+}"] \arrow[d] & \Delta.X \arrow[d, "p_X"] \\
\Gamma.\Sigma(A, B) \arrow[r, "="'] & \Gamma.\Sigma(A, B) \arrow[r, "v"'] & \Delta,
\end{tikzcd}$$

where the right hand square is a pullback. Now, giving a lift of the original square is the same as giving a lift for the left hand square. By the universal property of context extension, giving such a lift is equivalent to giving a term in $\mathsf{Tm}_{\Gamma(A,B)}(Xv)$ which is mapped to the term corresponding to u' by i^*. Since i^* is assumed to be an isomorphism, there is exactly one such term, which means that there is exactly one lift.

$(v) \to (i)$ Since right orthogonality classes are closed under composition (as it is immediate to verify), we know that i is left orthogonal to all weakenings. In particular, in the diagram

$$\begin{tikzcd} \Gamma.A.B \arrow[r, "="] \arrow[d, "i"'] & \Gamma.A.B \arrow[d] \\ \Gamma.\Sigma(A,B) \arrow[r] & \Gamma, \end{tikzcd}$$

we can find a diagonal lift $r\colon \Gamma.\Sigma(A,B) \to \Gamma.A.B$, so that $ri = \mathsf{id}$. It remains to show that $ir = \mathsf{id}$. In the square

$$\begin{tikzcd} \Gamma.A.B \arrow[r, "i"] \arrow[d, "i"'] & \Gamma.\Sigma(A,B) \arrow[d] \\ \Gamma.\Sigma(A,B) \arrow[r] & \Gamma \end{tikzcd}$$

both id and ir are diagonal lifts. By orthogonality of i, they are therefore equal, hence $ir = \mathsf{id}$, as required. □

Note that given a Σ-type structure, we can define a natural map $i\colon \Gamma.A.B \to \Gamma.\Sigma(A,B)$ using the universal property of context extension, and i satisfies condition (ii) of proposition 4.4 by construction, hence in particular it is an isomorphism by proposition 4.4 itself.

4.2 Π-types

The definition of Π-types is entirely analogous to that of Σ-types. We skip to the stable version directly.

Definition 4.5. A (stable) Π-*type structure* on a cwf $\mathcal{C}$ is given by:

- a natural transformation $\Pi\colon \mathsf{Ty}^{(2)} \to \mathsf{Ty}$;
- a natural isomorphism $\mathsf{Tm}_\Gamma(\Pi(A,B)) \cong \mathsf{Tm}_{\Gamma.A}(B)$ of functors $\mathcal{C}/\mathsf{Ty}^{(2)} \to \mathsf{Set}$.

Here applying the isomorphism left to right corresponds to application of a dependent function, while right to left is λ-abstraction. Again, the fact that these two maps compose to identities corresponds to the β and η rules of Π-types.

As for Σ-types, we can characterise the natural isomorphism in definition 4.5 without using terms. This time, however, we do not have a simple isomorphism of contexts as in the case of Σ-types, and not as many different characterisations.

Proposition 4.6. *Let* $\Pi\colon \mathsf{Ty}^{(2)} \to \mathsf{Ty}$ *be a natural transformation, and*

$$\epsilon\colon \Gamma.A.\Pi(A,B) \to \Gamma.A.B$$

a morphism over $\Gamma.A$*, natural in* $(\Gamma, A, B) \in \mathcal{C}/\mathsf{Ty}^{(2)}$*. The following are equivalent:*

(*i*) *the map* $\mathsf{Tm}_\Gamma(\Pi(A, B)) \to \mathsf{Tm}_{\Gamma.A}(B)$ *induced by* ϵ *is an isomorphism;*

(*ii*) *for all* $(\Delta, \sigma) \in \mathcal{C}/\Gamma$, ϵ *induces an isomorphism*

$$\mathcal{C}/\Gamma(\Delta, \Gamma.\Pi(A, B)) \cong \mathcal{C}/\Gamma.A(\Delta.A\sigma, \Gamma.A.B).$$

Proof. Entirely analogous to the proof of $(i) \leftrightarrow (ii)$ in proposition 4.4. □

Just like in the case of Σ-types, given a Π-type structure on $\mathcal{C}$, we can define a natural map $\epsilon \colon \Gamma.A.\Pi(A, B) \to \Gamma.A.B$ using the universal property of context extension, and proposition 4.6 applies.

4.3 Equality types

Equality types usually come in two flavours, which we will refer to as *extensional* and *intensional*. Extensional equality types are similar to Σ and Π types, and satisfy an analogous universal property. Intensional equality types are weaker, and do not satisfy a universal property in the usual categorical sense.

4.3.1 Extensional equality

Definition 4.7. An *extensional equality type structure* on a cwf $\mathcal{C}$ is given by a choice of types $\mathsf{Eq}_A \colon \mathsf{Ty}(\Gamma.A.A)$, natural in $(\Gamma, A) \in \mathcal{C}/\mathsf{Ty}$, such that the first projection

$$\coprod_{a,a' \in \mathsf{Tm}_\Gamma(A)} \mathsf{Tm}_\Gamma(\mathsf{Eq}_A(a, a')) \to \mathsf{Tm}_\Gamma(A), \tag{1}$$

is an isomorphism over $\mathsf{Tm}_\Gamma(A)^2$, where the map $\mathsf{Tm}_\Gamma(A) \to \mathsf{Tm}_\Gamma(A)^2$ is the diagonal.

Remark 4.8. The condition of definition 4.7 contains two statements: first, the first projection is a morphism over $\mathsf{Tm}_\Gamma(A)^2$, which means that the first and second projections are actually equal; second, both projections are isomorphisms. The two statements can be summarised by saying that for all terms $a, a' \in \mathsf{Tm}_\Gamma(A)$, there exists at most one term of type $\mathsf{Tm}_\Gamma(\mathsf{Eq}_A(a, a'))$, and it does exist if and only if $a = a'$. Note that the choice of using the first projection as the map is immaterial. In fact, there is at most one map (1) over $\mathsf{Tm}_\Gamma(A)^2$.

Remark 4.8 can be interpreted by saying that extensional equality types precisely reflect the external equality of terms into the logical structure of a cwf. An equality type is inhabited precisely when the corresponding terms are equal semantically, and there is at most one witness of equality.

Proposition 4.9. *Let* $\mathsf{Eq}_A \in \mathsf{Ty}(\Gamma.A.A)$, *natural in* (Γ, A), *and*

$$r \colon \Gamma.A \to \Gamma.A.A.\mathsf{Eq}_A$$

a natural morphism over $\Gamma.A.A$, *where the implicit map* $\Gamma.A \to \Gamma.A.A$ *is the variable* v_A. *The following are equivalent.*

(i) *r is an isomorphism;*

(ii) Eq_A *is an extensional equality type structure;*

(iii) *for all types $X \in \mathsf{Ty}(\Gamma.A.A.\mathsf{Eq}_A)$, the map*

$$r^* \colon \mathsf{Tm}_{\Gamma.A.A.\mathsf{Eq}_A}(X) \to \mathsf{Tm}_\Gamma(Xr)$$

is an isomorphism;

(iv) *r is left orthogonal to all display maps.*

Proof. (i) $\to$ (ii) By the universal property of context extension, r induces an isomorphism

$$\mathsf{Tm}_\Gamma(A) \to \coprod_{a,a' \in \mathsf{Tm}_\Gamma(A)} \mathsf{Tm}_\Gamma(\mathsf{Eq}_A(a,a'))$$

over $\mathsf{Tm}_\Gamma(A)^2$. The inverse of this isomorphism is therefore also a morphism over $\mathsf{Tm}_\Gamma(A)^2$. By remark 4.8 this inverse must be the first projection, which shows that Eq_A satisfies the condition of definition 4.7.

(ii) $\to$ (i) Let $(\Delta, \tau) \in \mathcal{C}/\Gamma.A.A$. Write $\tau = (\sigma, a, a')$, with $\sigma \colon \Delta \to \Gamma$, and $a, a' \in \mathsf{Tm}_\Delta(A\sigma)$. Since

$$\mathcal{C}/\Gamma.A.A(\Delta, \Gamma.A.A.\mathsf{Eq}_A) \cong \mathsf{Tm}_\Delta(\mathsf{Eq}_A \tau),$$

there is at most one such morphism, and there is one if and only if $a = a'$. By a similar argument, the same property is true for $\mathcal{C}/\Gamma.A.A(\Delta, \Gamma.A)$. It follows that the morphism

$$\mathcal{C}/\Gamma.A.A(\Delta, \Gamma.A) \to \mathcal{C}/\Gamma.A.A(\Delta, \Gamma.A.A.\mathsf{Eq}_A)$$

induced by r must be an isomorphism. By the Yoneda lemma, r itself is then an isomorphism.

The rest of the proof is completely analogous to that of proposition 4.4, and is left to the reader. □

4.3.2 Intensional equality

To define intensional equality, we weaken the definition of extensional equality types using their characterisation in terms of elimination properties (condition (iii) in proposition 4.9) or orthogonality (condition (iv)).

Definition 4.10. An *intensional equality type structure* on a cwf $\mathcal{C}$ is given by:

- a natural choice of types $\mathsf{Eq}_A \in \mathsf{Ty}(\Gamma.A.A)$;
- a natural map $r \colon \Gamma.A \to \Gamma.A.A.\mathsf{Eq}_A$ over $\Gamma.A.A$;
- for all $X \in \mathsf{Ty}(\Gamma.A.A.\mathsf{Eq}_A)$, a natural transformation

$$J \colon \mathsf{Tm}_{\Gamma.A}(Xr) \to \mathsf{Tm}_{\Gamma.A.A.\mathsf{Eq}}(X),$$

such that $r^*J = \mathsf{id}$.

Remark 4.11. Definition 4.10 is only correct if the cwf $\mathcal{C}$ has a Π-type structure. For a general cwf, definition 4.10 needs to be modified to include the so-called *Frobenius* condition, generalising the map J. Since we will not be interested in cwfs without Π-types, the simplified definition above is good enough for our purposes.

The naturality condition in definition 4.10 might not be entirely clear, so we will make explicit. Given a context morphism $\sigma\colon \Delta \to \Gamma$, it requires that the square

$$\begin{tikzcd}
\mathsf{Tm}_{\Delta.A\sigma}(X\sigma^{+++}r) \arrow[r, "J"] & \mathsf{Tm}_{\Delta.A\sigma.A\sigma.\mathsf{Eq}_{A\sigma}}(X\sigma^{+++}) \\
\mathsf{Tm}_{\Gamma.A}(Xr) \arrow[r, "J"'] \arrow[u, "(\sigma^+)^*"] & \mathsf{Tm}_{\Gamma.A.A.\mathsf{Eq}_A}(X) \arrow[u, "(\sigma^{+++})^*"']
\end{tikzcd}$$

commute, where we have implicitly used the fact that $r\sigma^+ = \sigma^{+++}r$.

Remark 4.12. The condition $r^*J = \mathsf{id}$ can be thought of as a β rule. The corresponding η rule, which is not assumed, would say that $Jr^* = \mathsf{id}$, and therefore imply that r^* is an isomorphism. Hence, by proposition 4.9, an intensional equality type structure satisfies the η rule if and only if it is an extensional equality type structure.

There is also a way to relate the definition of intensional equality types to an orthogonality condition corresponding to that of extensional equality types. Since intensional equality is weaker, we need to weaken the definition of orthogonality accordingly.

Definition 4.13. Let $f\colon A \to B$ and $g\colon X \to Y$ be morphisms in a category $\mathcal{A}$. We say that f has the *left lifting property* with respect to g (or equivalently that g has the right lifting property with respect to f) if for all commutative squares

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\
B \arrow[r, "v"'] & Y,
\end{tikzcd}$$

there exists a *diagonal lift*, i.e. a morphism $\ell\colon B \to X$ that makes both triangles commute:

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\
B \arrow[r, "v"'] \arrow[ur, "\ell"] & Y.
\end{tikzcd}$$

Definition 4.13 is virtually identical to definition 4.3, with the only difference being that the diagonal lift is not required to be unique. In particular, if f is left orthogonal to g, then it also has the left lifting property with respect to g. For this reason, one can also say that f is *weakly left orthogonal* to g to mean that it has the left lifting property with respect to g (and similarly for right).

Proposition 4.14. *Suppose $\mathcal{C}$ is equipped with an intensional equality type structure. Then for all $A \in \mathsf{Ty}(\Gamma)$, the morphism $r\colon \Gamma.A \to \Gamma.A.A.\mathsf{Eq}_A$ has the left lifting property with respect to all display maps.*

Proof. Let $\Delta \in \mathcal{C}$, $X \in \mathsf{Ty}(\Delta)$, and suppose given a commutative square

$$\begin{tikzcd} \Gamma.A \arrow[r, "u"] \arrow[d, "r"'] & \Delta.X \arrow[d, "p_X"] \\ \Gamma.A.A.\mathsf{Eq}_A \arrow[r, "v"'] & \Delta. \end{tikzcd}$$

By the universal property of context extension, there is a term $d \in \mathsf{Tm}_{\Gamma.A}(Xvr)$ such that $u = (vr, d)$. It is now easy to verify that $(v, Jd)\colon \Gamma.A.A.\mathsf{Eq}_A \to \Delta.X$ is a diagonal lift. □

4.4 Universes

A universe is, roughly speaking, a type having types as elements. Let us begin with a very weak (and general) definition.

Definition 4.15. A *universe* in a cwf $\mathcal{C}$ is a context $\mathcal{U}$, together with a type $\mathsf{El} \in \mathsf{Ty}(\mathcal{U})$.

The idea is that we can use El to convert morphisms $A \in \mathcal{C}(\Gamma, \mathcal{U})$ into types $\mathsf{El}(A) \in \mathsf{Ty}(\Gamma)$. Of course, this does not mean that we can identify such morphisms with types, as the conversion function is not guaranteed to be an isomorphism. However, we can define a new set of types using $\mathcal{U}$:

$$\mathsf{Ty}^{\mathcal{U}}(\Gamma) = \mathcal{C}(\Gamma, \mathcal{U}),$$

and it is easy to see that this can be extended to a new cwf structure on $\mathcal{C}$, where the corresponding functor of terms is given by

$$\mathsf{Tm}^{\mathcal{U}}_{\Gamma}(A) = \mathsf{Tm}_{\Gamma}(\mathsf{El}(A)).$$

We we refer to $\mathsf{Ty}^{\mathcal{U}}$ as the presheaf of types *classified by* $\mathcal{U}$. We say that $\mathcal{U}$ *classifies all types* if the map $\mathsf{Ty}^{\mathcal{U}}(\Gamma) \to \mathsf{Ty}(\Gamma)$ is an isomorphism.

As a special case of universe, we can take a context $\mathcal{U}$ that is obtained from the empty context (i.e. the distinguished terminal object 1) of $\mathcal{C}$ by extending it with a type in $\mathsf{Ty}(1)$, which we will also denote $\mathcal{U}$.

Definition 4.16. A *small universe* is a type $\mathcal{U} \in \mathsf{Ty}(1)$, together with a type $\mathsf{El} \in \mathsf{Ty}(\mathcal{U}) = \mathsf{Ty}(1.\mathcal{U})$.

In particular, any small universe is a universe. In most cases, one cannot expect a small universe to classify all types, because it would in particular classify itself, which tends to be impossible for size reasons.

Universes can often be used to turn unstable type formers into stable ones. We will see an example of this in section 5.3.4.

4.5 Type formers in presheaf models

We now define type formers for a presheaf model $\widehat{\mathcal{I}}$.

Lemma 4.17. *For $\Gamma \in \widehat{\mathcal{I}}$ and $A \in \widehat{\mathcal{I}/\Gamma}$, there is a natural isomorphism of categories*

$$(\mathcal{I}/\Gamma)/A \cong \mathcal{I}/(\Gamma.A)$$

over $\mathcal{I}/\Gamma$.

Proof. Objects of $(\mathcal{I}/\Gamma)/A$ are triples (i, x, a), with $i \in I$, $x \in \Gamma(i)$, and $a \in A(i, x)$. Using lemma 3.4, an identical description holds for the objects of $\mathcal{I}/(\Gamma.A)$. This establishes a bijection between the objects, and it is easy to verify that it extends to morphisms, hence to an isomorphisms of categories. Naturality can be verified easily. □

Now let $\Gamma \in \widehat{\mathcal{I}}$ be a presheaf. If $A \in \mathsf{Ty}(\Gamma)$, then types over $\Gamma.A$ are by definition presheaves on $I/(\Gamma.A)$, so by lemma 4.17 we can regard them as presheaves on $(\mathcal{I}/\Gamma)/A$. This means that we if think of A as a *context* in the presheaf category $\mathcal{I}/\Gamma$, then B can correspondingly be thought of as a type over that context.

This sort of *relativisation* procedure allows us to define most type formers for the special case of the empty context, as long as we define it for arbitrary presheaf categories. To extend a construction to the general case of a context $\Gamma \in \widehat{\mathcal{I}}$, we can simply work in $\widehat{\mathcal{I}/\Gamma}$ and apply the construction there.

Unfortunately, this is not quite enough, since there is no guarantee that a construction obtained this way is stable, so stability need to be checked separately.

The definition of Σ-types in $\widehat{\mathcal{I}}$ is particularly simple in the empty context. Given a type $A \in \mathsf{Ty}(1)$, and $B \in \mathsf{Ty}(1.A)$, we can think of A as a context, and define $\Sigma(A, B)$ to simply be $A.B$, which is clearly $\mathcal{V}$-small, hence can be regarded as a type over the empty context. Now we can extend this definition to an arbitrary context as explained above.

Making everything explicit, if $\Gamma \in \widehat{\mathcal{I}}$, $A \in \mathsf{Ty}(\Gamma)$ and $B \in \mathsf{Ty}(\Gamma.A)$, we define:

$$\Sigma(A, B)(i, x) = \coprod_{a \in A(i,x)} B(i, x, a).$$

Note that for the purpose of showing that the definition of the Σ-type structure is stable, it is actually important to give an explicit definition, or at least one that does not rely on uniqueness up to isomorphism, since stability is not preserved across isomorphisms. Fortunately, it is very easy to directly check stability in this case.

There is an obvious map $\iota\colon \Gamma.\Sigma(A,B) \to \Gamma.A.B$. In the representation where we think of A as a presheaf on $\mathcal{I}/\Gamma$, the map ι corresponds to the identity. Explicitly, ι is given by $\iota(x,(a,b)) = ((x,a),b)$. Since it simply amounts to rearranging brackets, it is clear that ι is a natural isomorphism. In conclusion, we get a Σ-type structure on $\widehat{\mathcal{I}}$.

To define Π-types, let us begin by reviewing exponentials in presheaf categories. Given presheaves A, B the *exponential* $[A,B]$ is a presheaf equipped with a natural isomorphism

$$\widehat{\mathcal{I}}(X,[A,B]) \cong \widehat{\mathcal{I}}(X \times A, B).$$

By Yoneda, $[A,B](i) \cong \widehat{\mathcal{I}}(I[i] \times A, B)$, and it is easy to see that, if we take this as the definition of $[A,B]$, the resulting presheaf does satisfy the universal property above.

Since Π-types can be thought of as types of dependent functions, it is natural to define them in terms of exponentials. We first give an unstable definition, denoted Π'. Let $\Gamma \in \widehat{\mathcal{I}}$ be any context, $A \in \mathsf{Ty}(\Gamma)$ and $B \in \mathsf{Ty}(\Gamma.A)$. We define $\Pi'(A,B)$ as the type obtained from the exponential $[A,\Sigma(A,B)]$ by taking only those functions that are the identity on A. More precisely, it is defined by the following pullback square in $\widehat{\mathcal{I}/\Gamma}$:

$$\begin{tikzcd} \Pi'(A,B) \arrow[r] \arrow[d] & {[A,\Sigma(A,B)]} \arrow[d] \\ 1 \arrow[r, "\mathsf{id}"'] & {[A,A]}, \end{tikzcd} \tag{2}$$

where the bottom horizontal map corresponds to the identity $A \to A$ through the universal property of the exponential.

Lemma 4.18. *Let $\mathcal{J} = \mathcal{I}/\Gamma$. There is a natural isomorphism*

$$\widehat{\mathcal{J}}(X,\Pi'(A,B)) \cong \widehat{\mathcal{J}}/A(X \times A, \Sigma(A,B)),$$

for $X \in \mathcal{J}$.

Proof. Apply the functor $\widehat{\mathcal{J}}(X,-)$ to the pullback square (2), and use the universal property of the exponentials involved. □

It then follows immediately that, for a fixed context Γ, Π' satisfies the universal property in the definition of Π-types. Given a morphism $\sigma\colon \Delta \to \Gamma$, it is easy to see that $\Pi'(A, B)\sigma$ also satisfies the universal property of the Π-type of $A\sigma$ and $B\sigma^+$, and therefore we get an isomorphism

$$\Pi'(A, B)\sigma \cong \Pi'(A\sigma, B\sigma^+).$$

To get a stable Π-type structure, we need to modify the definition of Π so that this isomorphism becomes an identity.

For $i \in \mathcal{I}$, and $x \in \Gamma(i)$, regard x as a morphism $\mathcal{I}[i] \to \Gamma$, and define

$$\Pi(A, B)(i, x) = \Pi'(Ax, Bx^+)(i, \mathsf{id}).$$

It is now easy to verify directly that this definition of Π is natural, and since $\Pi(A, B) \cong \Pi'(A, B)$, it also satisfies the same universal property. Therefore, we have defined a Π-type structure on $\widehat{\mathcal{I}}$.

Following a similar approach as for Σ-types, it is also possible to define an extensional equality type structure on $\widehat{\mathcal{I}}$, and directly prove its stability. We leave it as an exercise for the reader.

Finally, we define a universe in $\widehat{\mathcal{I}}$ that classifies all types. In other words, we want to construct a presheaf $\mathcal{U}$ that *represents* the functor Ty. Assume given such a $\mathcal{U}$, then by Yoneda:

$$\mathcal{U}(i) \cong \widehat{\mathcal{I}}(\mathcal{I}[i], \mathcal{U}) \cong \mathsf{Ty}(\mathcal{I}[i]),$$

which we can take as the definition of $\mathcal{U}$.

Lemma 4.19. *There is a natural isomorphism*

$$\widehat{\mathcal{I}}(\Gamma, \mathcal{U}) \cong \mathsf{Ty}(\Gamma),$$

for $\Gamma \in \widehat{\mathcal{I}}$.

Proof. Let $R\colon \mathsf{Cat}/\mathcal{I} \to \widehat{\mathcal{I}}$ be given by

$$R(\mathcal{E})(i) = \mathsf{Cat}/\mathcal{I}(\mathcal{I}/i, \mathcal{E}).$$

One can check that R is right adjoint to the functor which sends a presheaf to its category of elements. Observe that $\mathcal{U} \cong R(\mathcal{V}^{\mathrm{op}} \times \mathcal{I})$. Therefore

$$\widehat{\mathcal{I}}(\Gamma, \mathcal{U}) \cong \mathsf{Cat}/\mathcal{I}(\mathcal{I}/\Gamma, \mathcal{V}^{\mathrm{op}} \times \mathcal{I}) \cong \mathsf{Cat}(\mathcal{I}/\Gamma, \mathcal{V}^{\mathrm{op}}) = \mathsf{Ty}(\Gamma).$$

□

The type $\mathsf{El} \in \mathsf{Ty}(\mathcal{U})$, part of the universe structure for $\mathcal{U}$, can now be defined as the type corresponding to the identity $\mathcal{U} \to \mathcal{U}$ through the isomorphism of lemma 4.19. By unfolding the construction, we get that $\mathsf{El}_i(X) = X(i, \mathsf{id})$.

5 Simplicial sets

From now on, we will focus on the particular category $\mathbf{\Delta}$, defined below, and its category of presheaves $\widehat{\mathbf{\Delta}}$, whose objects are referred to as *simplicial sets*. From the point of view of the semantics of type theory, the motivation for considering simplicial sets can be traced back to the groupoid model.

Although one can use groupoids to model a good portion of homotopy type theory, univalent universes in the groupoid model contain only sets, which ultimately means that groupoids cannot be used to model many of the constructions arising from synthetic homotopy theory.

The naive solution to this problem is to generalise groupoids to *n-groupoids*, i.e. generalisations of groupoids containing n levels of morphisms (with 0-morphisms being objects), where the various algebraic laws only hold "weakly", i.e. up to the existence of a higher morphism. As it turns out, making such an idea precise involves a non-trivial amount of combinatorial complexity, and although there are certain approaches that follow this plan, ultimately a purely algebraic description of n-groupoids, including the limit case for $n = \infty$, is extremely hard to give and to work with.

For this reason, a different approach to higher groupoids, using simplicial sets, has been devised. As we will see, simplicial sets are essentially combinatorial descriptions of triangulated spaces of arbitrary dimensions. The key idea is to think of a simplicial set as the "carrier" of a groupoid, in the sense of the underlying family of sets, appropriately indexed, over which one would superimpose the groupoid structure and the corresponding laws.

However, instead of actual algebraic structure, we turn a simplicial set into something resembling a higher groupoid by simply asking for the *existence* of certain *lifts*, or, in other words, the ability to complete certain "partial diagrams" in the simplicial set. As it turns out, the mere existence of these lifts not only provides the required structure, but it also encodes the various weak laws that this structure has to satisfy. A simplicial set that satisfies such lifting properties is called a *Kan complex*.

The price we pay for this conceptual simplicity is that we lose the algebraic character of the structure, in the sense that operations are now better thought of as relations instead of functions. For example, the composition of two morphisms in such a groupoid is not uniquely defined, but we have to make an arbitrary choice to extract it. What we know is that a composite always exists, and furthermore we have some kind of "weak uniqueness", in the sense that any two composites are related by a higher morphism (in the appropriate sense), and this higher morphism is itself weakly unique in the same way.

Of course, all the general results about presheaf categories apply in particular to $\widehat{\mathbf{\Delta}}$, including the construction of the cwf structure, as well as the various type formers. However, in order to implement the idea of modelling higher groupoids using simplicial sets, we have to restrict the types of the cwf structure to *Kan fibrations*, i.e. those relative simplicial sets that do satisfy the lifting property mentioned above.

Consequently, we have to make sure that the type formers we have defined, namely Σ- and Π-types, restrict correctly to Kan fibration. Furthermore, we have to give a different definition of equality, which is specific to simplicial sets, and is *not* extensions, in order to have a chance at constructing a univalent universe. The universe itself is in fact relatively easy to define, but proving that it is a Kan complex is more involved. Interestingly, we will obtain univalence as an immediate by-product of that proof.

In the following, for n a natural number, we will denote by $[n]$ the ordered set of natural numbers *less or equal* to n. We will often regard $[n]$ as a category, with a morphism $i \to j$ being simply the assertion that $i \leq j$. When partially ordered sets are regarded as categories, monotone (increasing) maps between them correspond to functors.

Definition 5.1. The *simplex category* $\mathbf{\Delta}$ has natural numbers as objects, and morphisms

$$\mathbf{\Delta}(n, m) = \mathsf{Cat}([n], [m]).$$

A *simplicial set* is a presheaf on $\mathbf{\Delta}$.

By construction, the assignment $n \mapsto [n]$ extends to a fully faithful functor $\mathbf{\Delta} \to \mathsf{Cat}$. Since Cat is cocomplete, this functor determines an adjunction between $\widehat{\mathbf{\Delta}}$ and Cat. We will denote the left adjoint (*realisation*) simply by $\tau \colon \widehat{\mathbf{\Delta}} \to \mathsf{Cat}$, and the right adjoint (*nerve*) by $N \colon \mathsf{Cat} \to \widehat{\mathbf{\Delta}}$.

There is an important geometric intuition underlying the definition of $\mathbf{\Delta}$. Every object n can be thought of as an abstraction of a *geometric simplex* of dimension n, i.e. the convex hull of $n + 1$ affinely independent points in some Euclidean space. Through this analogy, the object 0, 1, 2 and 3 of $\mathbf{\Delta}$ can be thought of as a point, a segment, a triangle and a tetrahedron respectively.

Remark 5.2. The idea sketched above can be made precise by defining a functor $\mathbf{\Delta} \to \mathsf{Top}$ into the category of topological spaces. It is a useful exercise (although not particularly relevant for our purposes in these notes) to construct such a functor explicitly. Since Top is cocomplete, this functor determines an adjunction between $\widehat{\mathbf{\Delta}}$ and Top. The left adjoint $\widehat{\mathbf{\Delta}} \to \mathsf{Top}$ is usually called *geometric realisation*, while the right adjoint maps a topological space into its so-called *singular simplicial set*.

We can distinguish two special kinds of morphisms in $\mathbf{\Delta}$: the injective ones, forming a wide[1] subcategory $\mathbf{\Delta}_+$, and the surjective ones, forming a wide subcategory $\mathbf{\Delta}_-$. Note that for any injective map $f \in \mathbf{\Delta}(n, m)$, one necessarily has $n \leq m$. We say therefore that injective maps "raise dimension". Similarly, surjective maps lower dimension.

[1] A wide subcategory is a subcategory containing all objects.

There are exactly $n+1$ injective morphisms in $\mathbf{\Delta}(n-1,n)$. In fact, for all $i \in [n]$, there is a unique injective map $d_i \in \mathbf{\Delta}(n-1,n)$ whose image does not contain i. The morphisms d_i are called *face maps*. Similarly, there are $n+1$ surjective morphisms in $\mathbf{\Delta}(n+1,n)$. For all $i \in [n]$, there is a unique surjective map $s_i \in \mathbf{\Delta}(n, n+1)$ such that $s_i(i) = s_i(i+1)$. The morphisms s_i are called *degeneracy maps*.

Proposition 5.3. *The following facts hold for the simplicial category $\mathbf{\Delta}$:*

- *Every morphism can be uniquely factored as a surjective morphism followed by an injective morphism.*
- *Every surjective morphism has a section.*
- *Every morphism can be written as a composition of face and degeneracy maps.*

The proof of proposition 5.3 is left as an exercise for the reader.

A consequence of proposition 5.3 is that a simplicial set X can be thought of as a collection of sets $X(n)$ for all natural numbers n, together with face maps $d_i^* \colon X(n) \to X(n-1)$ and degeneracy maps $s_i^* \colon X(n) \to X(n+1)$, subject to appropriate conditions. The elements of $X(n)$ are referred to as *n-simplices* of X. The face map d_i sends an n-simplex to its i-th face, while the degeneracy map s_i sends it to a "degenerate" copy of itself, which is an $(n+1)$-simplex where the i-th and $(i+1)$-th vertex are the same.

Remark 5.4. Degenerate simplices have a less compelling geometric interpretation, and are harder to visualise and to draw, but they are important for technical reasons. Roughly speaking, degenerate simplices make it possible to have a simplicial representation for maps that "collapse" dimensions, such as the trivial map from any space to a point.

It is possible to devise a version of the theory of simplicial sets that forgoes degeneracies entirely, by focusing on the category $\mathbf{\Delta}_+$ and its presheaves. Such objects are called *semi-simplicial sets* (or Δ-sets, by some authors). The resulting theory is workable, but much more technically complicated.

Definition 5.5. Let $x \in X(n)$ be a simplex of a simplicial set X. We say that x is non-degenerate if for all surjective maps $\sigma \in \mathbf{\Delta}(k,n)$, and all simplices $y \in X(k)$, if $x = y\sigma$, then $\sigma = \mathsf{id}$.

Of course, a *degenerate* simplex is one that is not non-degenerate. Explicitly, x is degenerate if it can be written in the form $y\sigma$, with σ surjective and non-identity.

Proposition 5.6. *Let X be a simplicial set. Any simplex of X can be uniquely written in the form $y\sigma$, where y is non-degenerate and σ is surjective.*

Proof. Any n-simplex x can of course be written as $x(\mathsf{id})$, and id is surjective. Therefore, there is a minimal k such that $x = y\sigma$, for some k-simplex y, and $\sigma \in \mathbf{\Delta}_-(n,k)$. Minimality of k implies that y is non-degenerate.

As for uniqueness, let $x = z\tau$, with z a non-degenerate h-simplex and τ surjective. Since σ is surjective, it has a section $\theta \in \mathbf{\Delta}_+(k,n)$. We can factor $\tau\theta = \theta'\tau'$, with $\tau' \in \mathbf{\Delta}_-(k,m)$ and $\theta' \in \mathbf{\Delta}_+(m,h)$. Now, $y = y\sigma\theta = x\theta = z\tau\theta = z\theta'\tau'$, but y is non-denegerate, hence $\tau' = \mathsf{id}$, and in particular $m = k$. It follows that $k \leq h$. By reversing the roles of y and z in the above argument, we also get that $h \leq h$, and therefore $k = h$. But then $\theta' = \mathsf{id}$, hence $y = z$ and $\sigma = \tau$. $\square$

To simplify the notation when working with explicit low-dimensional simplices, we will employ the convention of denoting a map $f \in \mathbf{\Delta}(n,m)$ with the ordered tuple of its values. For example $02 \in \mathbf{\Delta}(1,3)$ is the map f such that $f(0) = 0$ and $f(1) = 2$. Given a 3-simplex x in a simplicial set X, $x(02)$ then denotes the 1-simplex xf.

5.1 Boundaries, Horns, Kan fibrations

In order to establish the lifting properties mentioned in the introduction to this section, we need to characterise certain elementary shapes within simplicial sets.

The simplest kind of shape is the abstract *n-simplex*, which we simply define as the representable $\mathbf{\Delta}[n]$. Of course, we think of $\mathbf{\Delta}[n]$ as a copy of the object n of $\mathbf{\Delta}$, living in the category of simplicial sets. Therefore, we can visualise it as its namesake n-dimensional simplex. Its elements correspond to (possibly degenerate) k-dimensional subsimplices. Because of degeneracies, k can be larger than n.

The next shape is the *n-boundary* $\partial\mathbf{\Delta}[n]$, which we define as the union of all the face maps $\mathbf{\Delta}[n-1] \to \mathbf{\Delta}[n]$. More precisely, consider the map

$$\coprod_{i=0}^{n} \mathbf{\Delta}[n-1] \to \mathbf{\Delta}[n],$$

determined by all the face maps, and factor it

$$\coprod_{i=0}^{n} \mathbf{\Delta}[n-1] \twoheadrightarrow \partial\mathbf{\Delta}[n] \rightarrowtail \mathbf{\Delta}[n],$$

as an epimorphism followed by a monomorphism (this can be done by taking the image levelwise for all $j \in \mathbf{\Delta}$). The resulting monomorphism $\partial\mathbf{\Delta}[n] \to \mathbf{\Delta}[n]$ is the *canonical inclusion of the n-boundary into the n-simplex*. Geometrically, $\partial\mathbf{\Delta}[n]$ is the boundary of the n-simplex, topologically equivalent to an $(n-1)$-sphere. Note that, for this definition to make sense, we must have $n > 0$. However, we extend the definition to $n = 0$ by setting $\partial\mathbf{\Delta}[0] = 0$, the empty simplicial set.

The final basic shape we are going to consider is the horn, given by the union of all the faces of a simplex *except one*. Similarly to how we defined the boundary, fix a natural number $n > 0$, and an index k between 0 and n, and consider the factorisation

$$\coprod_{i \neq k} \mathbf{\Delta}[n-1] \twoheadrightarrow \mathbf{\Lambda}^k[n] \rightarrowtail \mathbf{\Delta}[n]$$

of the map induced by face maps into an epimorphism followed a monomorphism. The resulting map $\mathbf{\Lambda}^k[n] \to \mathbf{\Delta}[n]$ is the *canonical inclusion of the* (k, n)*-horn into the* n*-simplex*.

We are now ready to define Kan fibrations, which, as mentioned in the introduction, will play a crucial role in the definition of the cwf structure on simplicial sets.

Definition 5.7. A *Kan fibration* is a map of simplicial sets that has the right lifting property with respect to all horn inclusions. A *Kan complex* is a simplicial set X such that the unique map $X \to 1$ is a Kan fibration.

The key concept at play here is that of a Kan complex, and a Kan fibration is simply the canonical way of *relativising* the notion of Kan complex to a family of simplicial sets. To understand the significance of Kan complexes and Kan fibrations, it is useful to keep in mind the following characterisations of horns and boundaries, whose proofs are left as exercises for the reader.

Lemma 5.8. *The* n*-boundary can be expressed as a coequaliser:*

$$\coprod_{i<j} \mathbf{\Delta}[n-2] \underset{v}{\overset{u}{\rightrightarrows}} \coprod_{i} \mathbf{\Delta}[n-1] \to \partial\mathbf{\Delta}[n],$$

where u *and* v *are characterised by the commutativity of the following squares:*

$$\begin{tikzcd}
\mathbf{\Delta}[n-2] \arrow[r, "d_{j-1}"] \arrow[d, "\ell_{i,j}"'] & \mathbf{\Delta}[n-1] \arrow[d, "\ell_i"] \\
\coprod_{i'<j'} \mathbf{\Delta}[n-2] \arrow[r, "u"'] & \coprod_{i'} \mathbf{\Delta}[n-1],
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{\Delta}[n-2] \arrow[r, "d_{i}"] \arrow[d, "\ell_{i,j}"'] & \mathbf{\Delta}[n-1] \arrow[d, "\ell_j"] \\
\coprod_{i'<j'} \mathbf{\Delta}[n-2] \arrow[r, "v"'] & \coprod_{i'} \mathbf{\Delta}[n-1],
\end{tikzcd}$$

and ℓ_i*,* $\ell_{i,j}$ *denote the canonical inclusions into the two coproducts.*

Lemma 5.9. *The* (k, n)*-horn can be expressed as a coequaliser:*

$$\coprod_{\substack{i<j \\ i,j \neq k}} \mathbf{\Delta}[n-2] \rightrightarrows \coprod_{i \neq k} \mathbf{\Delta}[n-1] \to \mathbf{\Lambda}^k[n],$$

where the maps are defined similarly to those of lemma 5.8.

Lemma 5.9 allows us to define a (k, n)-horn in a simplicial set X, i.e. a map $\mathbf{\Lambda}^k[n] \to X$, by specifying n simplices in X of dimension $n-1$, subject to conditions corresponding to the maps of the coequaliser. For example, to define a $(1,2)$-horn in X, we simply give two 1-simplices f, g of X, subject to the condition $f(1) = g(0)$.

If we focus on the 0- and 1-dimensional simplices of a simplicial set X, we can observe that they form a directed graph, where the vertices are exactly the 0-simplices, and the edges x to y are those 1-simplices f such that $f(0) = x$ and $f(1) = y$.

Now let us assume that X is a Kan complex. Two "composable" 1-simplices f and g (i.e. satisfying $f(1) = g(0)$) uniquely determine a map $[f, g] \colon \mathbf{\Lambda}^1[2] \to X$, as we have observed above. Therefore, the lifting condition for X says that there exists a 2-simplex $\alpha \colon \mathbf{\Delta}[2] \to X$ extending $[f, g]$. In particular, we get a 1-simplex $h = \alpha(02) \colon \mathbf{\Delta}[1] \to X$ between $f(0)$ and $g(2)$, which we can think of as *a* composition of f and g.

As briefly mentioned in the introduction, a Kan complex does not come equipped with specified composites for such 1-simplices, but only with a relation that determines when a simplex like h is indeed *a* composition of f and g. Namely, this is the case exactly when there is a 2-simplex α whose faces are exactly f, g and h.

The $(1, 2)$-horn lifting condition can therefore be thought of as stating that every composable pair of edges in X has a composition.

We have seen that the Yoneda lemma implies that every presheaf can be written as a colimit of representables. However, in the case of simplicial set, we can be more precise, and obtain a decomposition of any monomorphism as a colimit of a chain of "cell attachments", i.e. pushouts along (coproducts of) boundary inclusions.

Lemma 5.10. *Let $i \colon A \to B$ be a monomorphism of simplicial sets. There is a chain of morphisms $B_n \to B_{n+1}$, indexed over the natural numbers, such that:*

- $B_0 = A$;
- $\operatorname*{colim}_n B_n \cong B$;
- *i corresponds to the induced map $A \to \operatorname*{colim}_n B_n$;*
- *for all $n \geq 1$, there is a pushout diagram*

$$\begin{tikzcd}
\coprod_{x \in I_n} \partial\mathbf{\Delta}[n] \arrow[r] \arrow[d] & B_{n-1} \arrow[d] \\
\coprod_{x \in I_n} \mathbf{\Delta}[n] \arrow[r] & B_n,
\end{tikzcd}$$

where I_n is the set of non-degenerate n-simplices of B not contained in the image of A.

Proof. To simplify notation, we will regard A as a subfunctor of B. Let B_n be the subfunctor of B generated by the n-simplices of B and all the simplices of A. Then the first three properties are easy to verify, so we will only deal with the last one.

Fix a simplex $x \in I_n$. By Yoneda, we get a corresponding map $x\colon \mathbf{\Delta}[n] \to B$, and it easy to see that the restriction of x to $\partial\mathbf{\Delta}[n]$ lies in B_n. Taking the coproduct of all those maps x yields the top horizontal map in the pushout diagram above. The left vertical map is simply the coproduct of boundary inclusions.

One can quickly verify that the square commutes. To show that it is a pushout, we proceed by fixing an object $k \in \mathbf{\Delta}$, and proving that the corresponding diagram in Set

$$\coprod_{x\in I_n} \partial\mathbf{\Delta}[n](k) \rightrightarrows B_{n-1}(k) \amalg \coprod_{x\in I_n} \mathbf{\Delta}(k,n) \xrightarrow{\rho} B_n(k)$$

is a coequaliser.

First, observe that every n-simplex of B_n is either in B_{n-1} or non-degenerate. Therefore ρ is surjective.

Next, take any two elements in the domain of ρ which are mapped to the same element of $B_n(k)$. We have to show that they are identified in the coequaliser. We distinguish three cases.

If they both belong to $B_{n-1}(k)$, then they are equal, since the map $B_{n-1}(k) \to B_n(k)$ is an inclusion.

If $\rho(x,\theta) = b$, with $b \in B_{n-1}(k)$, write $b = b'\sigma$, for some non-degenerate $b' \in B_{n-1}(k')$, and σ surjective. If θ is also surjective, then by the uniqueness part of proposition 5.6 we have that $x = b'$, which is impossible, since $x \in I_n$, while b is k'-dimensional, and $k' < n$. Therefore θ is not surjective, hence $\theta \in \partial\mathbf{\Delta}[n](k)$, which implies that (x,θ) and b are identified in the coequaliser.

Finally, assume $\rho(x,\theta) = \rho(x',\theta')$, with $x, x' \in I_n$ and $\theta, \theta' \in \mathbf{\Delta}(k,n)$. If $x\theta \in B_{n-1}(k)$, then we are done by the previous step. Therefore, we can assume that θ is surjective. Similarly, θ' can be taken to be surjective. But then $x = x'$ and $\theta = \theta'$ by the uniqueness part of proposition 5.6 again. $\square$

5.2 Types and universes of Kan fibrations

The category $\widehat{\mathbf{\Delta}}$ of simplicial sets is a presheaf category, and therefore it comes equipped with a canonical cwf structure supporting Σ-types, Π-types, extensional equality types and a universe classifying all types, which we constructed in sections 3.4 and 4.5.

Let us denote by $\mathsf{Ty}'\colon \widehat{\mathbf{\Delta}}^{\mathrm{op}} \to \mathsf{Set}$ the functor of types, part of the cwf structure on $\widehat{\mathbf{\Delta}}$ as a presheaf category. Define a new functor $\mathsf{Ty}\colon \widehat{\mathbf{\Delta}}^{\mathrm{op}} \to \mathsf{Set}$ by setting

$$\mathsf{Ty}(\Gamma) = \{A \in \mathsf{Ty}'(\Gamma) \mid \Gamma.A \xrightarrow{p_A} \Gamma \text{ is a Kan fibration}\}.$$

The fact that Ty is a well-defined functor is a consequence of the following proposition.

Proposition 5.11. *Let $A \in \mathsf{Ty}'(\Gamma)$ be a type over a context $\Gamma \in \widehat{\mathbf{\Delta}}$ such that $\Gamma.A \xrightarrow{p_A} \Gamma$ is a Kan fibration, and $\sigma\colon \Delta \to \Gamma$ any morphism. Then $\Delta.A\sigma \xrightarrow{p_{A\sigma}} \Delta$ is a Kan fibration.*

Proof. Consider a diagram

$$\begin{tikzcd}
\mathbf{\Lambda}^k[n] \arrow[r] \arrow[d] & \Delta.A\sigma \arrow[r, "\sigma^+"] \arrow[d, "p_{A\sigma}"'] & \Gamma.A \arrow[d, "p_A"] \\
\mathbf{\Delta}[n] \arrow[r] & \Delta \arrow[r, "\sigma"'] & \Gamma.
\end{tikzcd}$$

We know that the right square is a pullback by lemma 3.2, and the right vertical map is a fibration by assumption. Therefore, there exists a lift $\ell\colon \mathbf{\Delta}[n] \to \Gamma.A$ for the outer rectangle. Using the universal property of the pullback, we get an induced map $\ell'\colon \mathbf{\Delta}[n] \to \Delta.A\sigma$, and it is easy to verify that ℓ' is a diagonal lift for the left square. $\square$

Types in $\mathsf{Ty}(\Gamma)$ will be called *fibrant* types. The rest of the cwf structure on $\widehat{\mathbf{\Delta}}$ can now be transported to fibrant types in a straightforward way. For example, terms of a fibrant type A are simply defined to be terms of the underlying type of A.

Now, let $\mathcal{U}'$ be the Hofmann-Streicher universe for $\widehat{\mathbf{\Delta}}$. Recall that $\mathcal{U}'$ is defined so that $\widehat{\mathbf{\Delta}}(\Gamma, \mathcal{U}') \cong \mathsf{Ty}'(\Gamma)$. Let us now apply the same construction using Ty instead of Ty'. Explicitly, define a simplicial set $\mathcal{U}$ by:

$$\mathcal{U}(n) = \mathsf{Ty}(\mathbf{\Delta}[n]).$$

It is clear that $\mathcal{U}$ is a subfunctor of $\mathcal{U}'$. In particular, we can regard $\widehat{\mathbf{\Delta}}(\Gamma, \mathcal{U})$ as a subset of $\widehat{\mathbf{\Delta}}(\Gamma, \mathcal{U}')$, and in fact the following proposition implies that it is exactly the subset of those morphisms corresponding to fibrant types over Γ.

Remark 5.12. Note that the definition of $\mathcal{U}$ implies that an element $A \in \mathcal{U}'(n)$ belongs to $\mathcal{U}(n)$ if and only if the corresponding type $\mathsf{El}(A) \in \mathsf{Ty}'(\mathbf{\Delta}[n])$ is fibrant.

Proposition 5.13. *Let $A\colon \widehat{\mathbf{\Delta}}(\Gamma, \mathcal{U}')$. Then A factors through $\mathcal{U}$ if and only if* $\mathsf{El}(A) \in \mathsf{Ty}(\Gamma)$.

Proof. First assume that $\mathsf{El}(A) \in \mathsf{Ty}(\Gamma)$, and fix $n \in \mathbf{\Delta}$ and $x \in \Gamma(n)$. Then $\mathsf{El}(Ax) = \mathsf{El}(A)x \in \mathsf{Ty}(\mathbf{\Delta}[n])$ by proposition 5.11, hence $A(x) \in \mathcal{U}(n)$ by remark 5.12.

Conversely, suppose that A factors through $\mathcal{U}$, and consider a lifting problem

$$\begin{tikzcd} \mathbf{\Lambda}^k[n] \arrow[r, "u"] \arrow[d] & \Gamma.\mathsf{El}(A) \arrow[d] \\ \mathbf{\Delta}[n] \arrow[r, "v"'] & \Gamma. \end{tikzcd}$$

By lemma 3.2, we can insert a pullback square in the diagram as follows:

$$\begin{tikzcd} \mathbf{\Lambda}^k[n] \arrow[r] \arrow[d] & \mathbf{\Delta}[n].\mathsf{El}(Av) \arrow[r, "u"] \arrow[d] & \Gamma.\mathsf{El}(A) \arrow[d] \\ \mathbf{\Delta}[n] \arrow[r, "="'] & \mathbf{\Delta}[n] \arrow[r, "v"'] & \Gamma, \end{tikzcd}$$

and the middle vertical map is a fibration by remark 5.12. Therefore, we get a lift for the left square, and composing with u yields a lift for the original square. □

5.3 Type formers and Kan fibrations

In this section we will construct the various type formers defined in section 4 in the cwf of simplicial sets and Kan fibrations, which we have set up in section 5.2.

5.3.1 Open prisms

We begin with some preliminary results of combinatorial nature, which will prove very useful in the following sections.

If n is a natural number, let the (k-oriented) *open n-prism* $P^k[n]$ be the simplicial set defined by the pushout

$$\begin{tikzcd} \partial\mathbf{\Delta}[n] \arrow[r, "\mathsf{id}\times(k)"] \arrow[d] & \partial\mathbf{\Delta}[n] \times \mathbf{\Delta}[1] \arrow[d] \\ \mathbf{\Delta}[n] \arrow[r] & P^k[n]. \end{tikzcd}$$

Geometrically, $P^0[n]$ is a subspace of the prism $\mathbf{\Delta}[n] \times \mathbf{\Delta}[1]$ consisting of the base of the prism $\mathbf{\Delta}[n] \times \{0\}$, together with all the side faces. The top face, as well as the interior of the prism, are missing.

In fact, there is a canonical map $i\colon P^k[n] \to \mathbf{\Delta}[n] \times \mathbf{\Delta}[1]$, called an *open prism inclusion*, induced by the inclusions $\partial\mathbf{\Delta}[n] \to \mathbf{\Delta}[n]$ and $\mathbf{\Delta}[0] \xrightarrow{(k)} \mathbf{\Delta}[1]$, and it is easy to see that i is a monomorphism.

More generally, we can start with an arbitrary monomorphism $j\colon A \to B$ of simplicial sets, and obtain a similarly defined monomorphism

$$i\colon B \amalg_A (A \times \mathbf{\Delta}[1]) \to B \times \mathbf{\Delta}[1],$$

which we will refer to as a *generalised open prism inclusion*.

Prism inclusions and their generalisations are of fundamental technical importance in the study of Kan fibrations, since the former play the same role of horn inclusion in the characterisation of the latter.

Lemma 5.14. *Let $p\colon Y \to X$ an arbitrary map of simplicial sets, and let $\mathcal{L}$ be the class of maps that have the left lifting property with respect to p.*

(i) *If $f_i\colon A_i \to B_i$ is a family of maps in $\mathcal{L}$, then the coproduct map $\coprod_i A_i \to \coprod_i B_i$ belongs to $\mathcal{L}$.*

(ii) *If $f\colon A \to B$ is in $\mathcal{L}$, and*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r] & D \end{tikzcd}$$

is a pushout square, then $g \in \mathcal{L}$.

(iii) *If $f_n\colon A_n \to A_{n+1}$ is a chain of maps in $\mathcal{L}$, indexed by the natural numbers, their* infinite composition, *i.e. the canonical map $A_0 \to \operatorname{colim}_n A_n$, is also in $\mathcal{L}$.*

(iv) *Let $f\colon A \to B$ be a retract of $g\colon C \to D$, i.e. suppose that there exists a diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[r] \arrow[d, "g"'] & A \arrow[d, "f"] \\ B \arrow[r] & D \arrow[r] & B, \end{tikzcd}$$

where the horizontal maps $A \to A$ and $B \to B$ are both identities. Then $g \in \mathcal{L}$ implies $f \in \mathcal{L}$.

The proof of lemma 5.14 is left as an exercise for the reader.

Remark 5.15. Dualising lemma 5.14 yields a statement about the class of maps defined by a right lifting property, such as Kan fibrations. In particular, Kan fibrations are closed under pullbacks (cf. proposition 5.11), finite compositions and retracts.

Proposition 5.16. *Let $p\colon Y \to X$ be a map of simplicial sets. The following are equivalent:*

(i) *p is a Kan fibration;*

(ii) *p has the right lifting property with respect to all open prism inclusions;*

(iii) *p has the right lifting property with respect to all generalised open prism inclusions.*

Proof. $(i) \to (ii)$ By lemma 5.14 it is enough to prove that open prism inclusions can be obtained as compositions of pushouts of horn inclusions, and by symmetry we can limit ourselves to 1-oriented open prism inclusions. For $0 \leq k \leq n$, let σ_k be the non-degenerate $(n+1)$-simplex of the prism $\mathbf{\Delta}[n] \times \mathbf{\Delta}[1]$ defined as the image through the nerve of the functor

$$\sigma_k \colon [n+1] \to [n] \times [1]$$
$$\sigma_k(i) = \begin{cases} (i, 0) & \text{for } i \leq k, \\ (i-1, 1) & \text{otherwise.} \end{cases}$$

Let A_k be the subfunctor of $\mathbf{\Delta}[n] \times \mathbf{\Delta}[1]$ generated by the open prism and the simplices σ_i for $i \leq k$. Then, by construction, the open prism inclusion $P^1[n] \to \mathbf{\Delta}[n] \times \mathbf{\Delta}[1]$ can be factored as a composition of inclusions $A_{k-1} \to A_k$, for $0 \leq k \leq n$, with the convention that $A_{-1} = P^1[n]$.

One can then check directly that there is a pullback square of inclusions:

$$\begin{tikzcd} \mathbf{\Lambda}^{k+1}[n+1] \arrow[r] \arrow[d] & A_{k-1} \arrow[d] \\ \mathbf{\Delta}[n+1] \arrow[r, "\sigma_k"'] & A_k. \end{tikzcd}$$

Since the bottom horizontal map and the right vertical map are jointly surjective, it follows that the square above is also a pushout, as required.

$(ii) \to (iii)$ Suppose that p has the right lifting property with respect to open prism inclusions. It then follows easily from the adjunction defining exponentials that the induced map

$$p' \colon [\mathbf{\Delta}[1], Y] \to [\mathbf{\Delta}[1], X] \times_X Y$$

has the right lifting property with respect to all boundary inclusions. By lemmas 5.10 and 5.14, p' has the right lifting property with respect to all monomorphisms, which, using the the exponential adjunction again, implies that p has the right lifting property with respect to generalised open prism inclusions.

$(iii) \to (i)$ One can show that, for $k < n$, the horn inclusion $i \colon \mathbf{\Lambda}^k[n] \to \mathbf{\Delta}[n]$ is a retract of the 0-oriented generalised open prism inclusion corresponding to i

itself. More explicitly, there is a diagram

$$\begin{tikzcd}
\mathbf{\Lambda}^k[n] \arrow[r] \arrow[d] & \mathbf{\Delta}[n] \amalg_{\mathbf{\Lambda}^k[n]} (\mathbf{\Lambda}^k[n] \times \mathbf{\Delta}[1]) \arrow[r] \arrow[d] & \mathbf{\Lambda}^k[n] \arrow[d] \\
\mathbf{\Delta}[n] \arrow[r, "\mathsf{id} \times (1)"'] & \mathbf{\Delta}[n] \times \mathbf{\Delta}[1] \arrow[r, "r"'] & \mathbf{\Delta}[n],
\end{tikzcd}$$

where the top maps are restrictions of the bottom maps, and r is the image through the nerve of the functor

$$r\colon [n] \times [1] \to [n]$$
$$r(i, b) = \begin{cases} k & \text{if } i \geq k \text{ and } b = 0 \\ i & \text{otherwise} \end{cases}$$

We leave the details of the proof to the reader. In the case $k > 0$ one can analogously show that $\mathbf{\Lambda}^k[n] \to \mathbf{\Delta}[n]$ is a retract of a 1-oriented generalised open prism inclusion. □

5.3.2 Σ-types

The case of Σ-types is particularly simple. The idea is to reuse the construction of Σ-types in general presheaf models. Since fibrant types are in particular types for the presheaf cwf structure, we automatically get a Σ-type structure satisfying the correct universal property and stability condition as soon as we prove that fibrancy is preserved by the Σ operation on types. This is achieved by the following result.

Proposition 5.17. *Let $\Gamma \in \widehat{\mathbf{\Delta}}$ be a context, and $A \in \mathsf{Ty}(\Gamma)$, $B \in \mathsf{Ty}(\Gamma.A)$ be fibrant types. Then $\Sigma(A, B)$ is fibrant.*

Proof. Since $\Gamma.\Sigma(A, B) \cong \Gamma.A.B$ over Γ, the map $\Gamma.\Sigma(A, B) \to \Gamma$ is isomorphic to the composition $\Gamma.A.B \to \Gamma.A \to \Gamma$, and therefore it is a Kan fibration by remark 5.15. □

5.3.3 Homotopy

The definition of the simplicial model of homotopy type theory is based on the "homotopical" character of simplicial sets, i.e. the ability of the category $\widehat{\mathbf{\Delta}}$ to act as a "model" for the homotopy theory of topological spaces.

Making such a statement precise is possible, but outside of the scope of these notes. It is, however, relatively easy, and already quite useful, to formulate a notion of homotopy between maps. This will be a key concept in the construction of intensional identity types and Π-types.

Definition 5.18. Let $f, g\colon X \to Y$ be maps of simplicial sets. A *homotopy* $h\colon f \sim g$ is a morphism

$$h\colon X \times \mathbf{\Delta}[1] \to Y,$$

such that $h \circ (\mathsf{id} \times (0)) = f$ and $h \circ (\mathsf{id} \times (1))$.

Definition 5.18 is a direct simplicial translation of the corresponding topological notion, where we are using the abstract 1-simplex to play the role of the topological interval. Similarly, we can define a notion of equivalence as an appropriately well-behaved "isomorphism up to homotopy".

Definition 5.19. Let $f\colon X \to Y$ be a map of simplicial sets. We say that f is a *strong homotopy equivalence* if there exists $g\colon Y \to X$, and homotopies $h\colon gf \sim \mathsf{id}_X$, $k\colon fg \sim \mathsf{id}_Y$, such that the square

$$\begin{tikzcd}
X \times \mathbf{\Delta}[1] \arrow[r, "f \times \mathsf{id}"] \arrow[d, "h"'] & Y \times \mathbf{\Delta}[1] \arrow[d, "k"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

commutes.

The importance of definition 5.19 lies in the following result.

Lemma 5.20. *Let $f\colon X \to Y$ be a monomorphism and a strong homotopy equivalence. Then f has the left lifting property with respect to fibrations.*

Proof. Let i be the generalised open prism inclusion corresponding to f. Let $g\colon Y \to X$ a homotopy inverse of f, and h, k the homotopies given by the definition of strong homotopy equivalence. We can construct a commutative square:

$$\begin{tikzcd}
Y \amalg_X (X \times \mathbf{\Delta}[1]) \arrow[r, "{[g,h]}"] \arrow[d, "i"'] & X \arrow[d, "f"] \\
Y \times \mathbf{\Delta}[1] \arrow[r, "k"'] & Y,
\end{tikzcd}$$

and it follows that f is a retract of i. Since i has the left lifting property with respect to fibrations by proposition 5.16, so does f by lemma 5.14. $\square$

5.3.4 Intensional equality types

Definition 5.21. Let $\Gamma \in \widehat{\mathbf{\Delta}}$, and $A \in \mathsf{Ty}(\Gamma)$ a fibrant type. Define the *path type* of A:

$$P(A) = \Pi(\mathbf{\Delta}[1], A),$$

where $\mathbf{\Delta}[1]$ is implicitly weakened to be a type in the context Γ.

By definition, $P(A)$ is a type over Γ. We now want to show that $P(A)$ can also be regarded as a type over $\Gamma.A.A$, and therefore be taken as the definition of an intensional type structure on $\widehat{\mathbf{\Delta}}$.

First, observe that $\Gamma.\Pi(\mathbf{\Delta}[0], A) \cong \Gamma.A$ over Γ. Since $\partial\mathbf{\Delta}[1] \cong \mathbf{\Delta}[0] \amalg \mathbf{\Delta}[0]$, it follows there is an isomorphism $\Gamma.\Pi(\partial\mathbf{\Delta}[1], A) \cong \Gamma.A.A$ over Γ. The boundary inclusion $\partial\mathbf{\Delta}[1] \to \mathbf{\Delta}[1]$ then determines a map with $\mathcal{V}$-small fibres $\Gamma.P(A) \to \Gamma.A.A$, and therefore a (not necessarily fibrant) type $\mathsf{Eq}_A \in \mathsf{Ty}(\Gamma.A.A)$. Our goal is now to show that Eq_A is in fact fibrant.

Let us consider a generalisation of generalised open prisms, defined for an arbitrary pair of morphisms $f\colon A \to B$ and $g\colon C \to D$. There is an obvious commutative square

$$\begin{tikzcd} A \times C \arrow[r] \arrow[d] & A \times D \arrow[d] \\ B \times C \arrow[r] & B \times D, \end{tikzcd}$$

which determines a map $f \widehat{\times} g\colon (B \times C) \amalg_{A \times C} (A \times D) \to B \times D$ from the pushout to the bottom right corner. We call $f \widehat{\times} g$ the *Leibniz product* of f and g, because it is reminiscent of the Leibniz rule for the derivative of a product.

Note that, using this language, a generalised open prism inclusion induced by a monomorphism $j\colon A \to B$ is simply the Leibniz product of j with an endpoint inclusion $\mathbf{\Delta}[0] \to \mathbf{\Delta}[1]$.

Leibniz product is a special case of the *Leibniz construction*. See [14] for a detailed exposition of its properties. In the following, we will only need the following.

Lemma 5.22. *Leibniz product is associative, i.e. given maps $f\colon A \to B$, $g\colon C \to D$, $h\colon E \to F$, there is a canonical isomorphism $(f \widehat{\times} g) \widehat{\times} h \cong f \widehat{\times} (g \widehat{\times} h)$ in the arrow category $\widehat{\mathbf{\Delta}}^{[1]}$.*

Proof. Consider the diagram X obtained from the cube

$$\begin{tikzcd}[row sep=small, column sep=small] & A \times D \times E \arrow[rr] \arrow[dd] & & B \times D \times E \arrow[dd] \\ A \times C \times E \arrow[ur] \arrow[rr] \arrow[dd] & & B \times C \times E \arrow[ur] \arrow[dd] & \\ & A \times D \times F \arrow[rr] & & B \times D \times F \\ A \times C \times F \arrow[ur] \arrow[rr] & & B \times C \times F \arrow[ur] & \end{tikzcd}$$

by removing the corner corresponding to $B \times D \times F$. A lengthy but straightforward diagram chasing argument shows that the domains of both $(f \widehat{\times} g) \widehat{\times} h$ and $f \widehat{\times} (g \widehat{\times} h)$ come equipped with a universal cocone from X such that the corresponding maps into $B \times D \times F$ are induced by the cube above using the universal property of the colimit.

The conclusion then follows directly from uniqueness of colimits up to isomorphism. $\square$

Proposition 5.23. *The map $p \colon \Gamma.P(A) \to \Gamma.A.A$ is a Kan fibration, hence Eq_A is a fibrant type.*

Proof. Let $f \colon X \to Y$ be an arbitrary map. Given a lifting problem

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \Gamma.\Pi(\mathbf{\Delta}[1], A) \arrow[d] \\ Y \arrow[r] & \Gamma.\Pi(\partial\mathbf{\Delta}[1], A), \end{tikzcd}$$

the universal property of Π-types implies that finding a lift in the above diagram is equivalent to finding a lift in the diagram

$$\begin{tikzcd} (X \times \mathbf{\Delta}[1]) \amalg_{X \times \partial\mathbf{\Delta}[1]} (Y \times \partial\mathbf{\Delta}[1]) \arrow[r] \arrow[d] & \Gamma.A \arrow[d] \\ Y \times \mathbf{\Delta}[1] \arrow[r] & \Gamma \end{tikzcd}$$

where the left vertical map is the Leibniz product of f and the boundary inclusion $j \colon \partial\mathbf{\Delta}[1] \to \mathbf{\Delta}[1]$.

To show that p is a fibration, it enough, by proposition 5.16, to show that p has the right lifting property with respect to generalised open prism inclusions. By the above argument, that is itself equivalent to showing that $\Gamma.A \to \Gamma$ has the right lifting property with respect to maps $v = u \widehat{\times} j$, where u is a generalised open prism inclusion.

Now, $u = (k) \widehat{\times} i$, where $(k) \colon \mathbf{\Delta}[0] \to \mathbf{\Delta}[1]$ is an endpoint inclusion and i is an arbitrary monomorphism. Therefore, lemma 5.22 implies that $v \cong (k) \widehat{\times} (i \widehat{\times} j)$, and it is easy to check that the Leibniz products of two monomorphisms is a monomorphism, hence v is itself a generalised open prism inclusion.

Since $\Gamma.A \to \Gamma$ is a Kan fibration, it has the right lifting property with respect to v, and therefore p has the right lifting property with respect to u, as claimed. $\square$

Now that we have an equality type, it remains to define the reflexivity map and the corresponding elimination property.

Define $r\colon \Gamma.A \to \Gamma.A.A.\mathsf{Eq}_A$ as the map induced on Π-types by the unique map $\mathbf{\Delta}[1] \to \mathbf{\Delta}[0]$, or in other words, the map corresponding to the projection

$$\Gamma.A \times \mathbf{\Delta}[1] \to \Gamma.A,$$

through the universal property of Π-types. Explicitly, for all $n \in \mathbf{\Delta}$, $x \in \Gamma(n)$ and $a \in A(x, a)$, we have $r(x, a) = (x, p)$, where $p\colon \mathbf{\Delta}[n] \times \mathbf{\Delta}[1] \to \mathbf{\Delta}[n].Ax$ is given by $p(u, t) = (u, au)$.

Proposition 5.24. *The map $r\colon \Gamma.A \to \Gamma.A.A.\mathsf{Eq}_A$ has the left lifting property with respect to fibrations.*

Proof. Let $c\colon \mathbf{\Delta}[1] \times \mathbf{\Delta}[1] \to \mathbf{\Delta}[1]$ be the image through the nerve of the functor

$$\mathsf{max}\colon [1] \times [1] \to [1].$$

Correspondingly, we get a map

$$\Gamma.\Pi(\mathbf{\Delta}[1], A) \times \mathbf{\Delta}[1] \times \mathbf{\Delta}[1] \xrightarrow{\mathsf{id} \times c} \Gamma.\Pi(\mathbf{\Delta}[1], A) \times \mathbf{\Delta}[1] \xrightarrow{\epsilon} \Gamma.A,$$

and hence a map

$$h\colon \Gamma.\Pi(\mathbf{\Delta}[1], A) \times \mathbf{\Delta}[1] \to \Gamma.\Pi(\mathbf{\Delta}[1], A).$$

Explicitly, for all $n \in \mathbf{\Delta}$, $x \in \Gamma(n)$, $p\colon \mathbf{\Delta}[n] \times \mathbf{\Delta}[1] \to \mathbf{\Delta}[n].Ax$, and $t \in \mathbf{\Delta}(n, 1)$, we have $h(x, p, t) = (x, q)$ where $q(u, s) = p(u, c(s, t))$.

Now let $\pi_1\colon \Gamma.A.A.\mathsf{Eq}_A \to \Gamma.A$ be the projection on the second A factor. Explicitly, $\pi_1(x, p) = (x, p(\mathsf{id}, 1))$. It is easy to verify that h is a homotopy between the identity and $\pi_1 \circ r$. Furthermore, $r \circ \pi_1 = \mathsf{id}_{\Gamma.A}$. Finally, the square

$$\begin{tikzcd}
\Gamma.A \times \mathbf{\Delta}[1] \arrow[r, "r \times \mathsf{id}"] \arrow[d] & \Gamma.A.A.\mathsf{Eq}_A \times \mathbf{\Delta}[1] \arrow[d, "h"] \\
\Gamma.A \arrow[r, "r"'] & \Gamma.A.A.\mathsf{Eq}_A
\end{tikzcd}$$

commutes, as one can verify by using the explicit expressions for the maps involved. In conclusion, r is a strong homotopy equivalence. But r is clearly a monomorphism, since it has a left inverse, therefore it has the left lifting property with respect to fibrations by lemma 5.20. $\square$

We will now complete the construction of the intensional equality type structure on $\widehat{\mathbf{\Delta}}$. We know that Eq and the reflexivity map are stable, since they are built using the stable Π-type structure of the presheaf model. The fact that the reflexivity map has the left lifting property with respect to display maps (proposition 5.24) can be used to construct an eliminator, but we need to be careful to ensure that the eliminator is stable.

Proposition 5.25. *The cwf of simplicial sets has an intensional equality type structure.*

Proof. We already have Eq and r, so it remains to define a natural map

$$J\colon \mathsf{Tm}_{\Gamma.A}(Xr) \to \mathsf{Tm}_{\Gamma.A.A.\mathsf{Eq}_A}(X),$$

for all contexts $\Gamma \in \widehat{\mathbf{\Delta}}$, $A \in \mathsf{Ty}(\Gamma)$, $X \in \mathsf{Ty}(\Gamma.A.A.\mathsf{Eq}_A)$, such that $r^*J = \mathsf{id}$.

The main idea is to use the universe $\mathcal{U}$ to construct a context Θ that "classifies" the possible parameters of J, i.e. the triples (A, X, d), with A, X as above, and $d \in \mathsf{Tm}_{\Gamma.A}(Xr)$. This means that there exists a universal such triple $(\mathfrak{A}, \mathfrak{X}, \mathfrak{d})$ in context Θ such that, for all contexts Γ, and triples (A, X, d), there exists a unique morphism $\phi_{A,X,d}\colon \Gamma \to \Theta$ with $A = \mathfrak{A}\phi_{A,X,d}$, $X = \mathfrak{X}\phi^{+++}_{A,X,d}$ and $d = \mathfrak{d}\phi^{+}_{A,X,d}$.

We can do this in stages, one for each of the three parameters. The first stage is $\Theta_0 = \mathcal{U}$, and $\mathfrak{A}$ is simply El, which is a universal fibrant type by proposition 5.13. For the next stage, we take the exponential $\Theta_1 = [\Pi(\mathbf{\Delta}[1], \mathfrak{A}), \Theta_0 \times \mathcal{U}]$ in the category $\widehat{\mathbf{\Delta}}/\Theta_0$, where the map $\Theta_1 \times \mathcal{U} \to \mathcal{U}$ is the second projection.

The type $\mathfrak{X}$ is defined to be the one corresponding to the composition

$$\Theta_1.\mathfrak{A}.\mathfrak{A}.\mathsf{Eq}_{\mathfrak{A}} \cong \Theta_1 \times_{\Theta_0} \Theta_0.\Pi(\mathbf{\Delta}[1], \mathfrak{A}) \xrightarrow{\epsilon} \Theta_0 \times \mathcal{U} \to \mathcal{U},$$

where ϵ is the evaluation map.

The final step Θ is just the context extension of Θ_1 with the type $\Pi(\mathfrak{A}, \mathfrak{X}r)$, and we define $\mathfrak{d}$ to be the corresponding variable. Thanks to the universal property of Π-types, we can regard $\mathfrak{d}$ as a morphism $\Theta.\mathfrak{A} \to \Theta.\mathfrak{X}r$ over Θ.

Now consider the "universal" lifting problem:

$$\begin{tikzcd}
\Theta.\mathfrak{A} \arrow[r] \arrow[d, "r"'] & \Theta.\mathfrak{A}.\mathfrak{A}.\mathsf{Eq}_{\mathfrak{A}}.\mathfrak{X} \arrow[d] \\
\Theta.\mathfrak{A}.\mathfrak{A}.\mathsf{Eq}_{\mathfrak{A}} \arrow[r, "="'] & \Theta.\mathfrak{A}.\mathfrak{A}.\mathsf{Eq}_{\mathfrak{A}},
\end{tikzcd}$$

where the top horizontal map is the composition

$$\Theta.\mathfrak{A} \xrightarrow{\mathfrak{d}} \Theta.\mathfrak{X}r \xrightarrow{r^+} \Theta.\mathfrak{A}.\mathfrak{A}.\mathsf{Eq}_{\mathfrak{A}}.\mathfrak{X},$$

and fix a lift, which we can regard as a term $J_0 \in \mathsf{Tm}_{\Theta.\mathfrak{A}.\mathfrak{A}.\mathsf{Eq}_{\mathfrak{A}}}(\mathfrak{X})$ satisfying $J_0 r = \mathfrak{d}$.

Now we can define J as $J(d) = J_0\phi^{+++}_{A,X,d}$.

The fact that J is a valid eliminator follows from naturality of r, since we have $J(d)r = J_0\phi^{+++}_{A,X,d}r = J_0 r\phi^{+}_{A,X,d} = \mathfrak{d}\phi^{+}_{A,X,d} = d$.

Furthermore, J is stable. In fact, given any morphism $\sigma\colon \Delta \to \Gamma$, we have $\phi_{A\sigma, X\sigma^{+++}, d\sigma^+} = \phi_{A,X,d}\sigma$, and therefore

$$J(d\sigma^+) = J_0\phi^{+++}_{A\sigma, X\sigma^{+++}, d\sigma^+} = J_0\phi^{+++}_{A,X,d}\sigma^{+++} = J(d)\sigma^{+++}.$$

$\square$

5.3.5 Π-types

The goal of this section is to construct a Π-type structure on the cwf of simplicial sets and Kan fibrations. As for Σ-types, we are going to reuse the Π-type structure on a general presheaf category, and therefore we only need to verify that Kan fibrations are closed under Π-types. More precisely, we are going to prove the following:

Proposition 5.26. *Let $\Gamma \in \widehat{\mathbf{\Delta}}$ be a context, $A \in \mathsf{Ty}(\Gamma)$, and $B \in \mathsf{Ty}(\Gamma.A)$ fibrant types. Then $\Pi(A, B)$ is fibrant.*

The proof of proposition 5.26 follows the general approach of [9], and is comprised of two main steps:

1. Use the universal property of Π-types to reduce the statement to a closure property of the class of maps that have the left lifting property with respect to fibrations.
2. Prove the above closure property using the auxiliary notion of strong homotopy equivalence (definition 5.19).

In fact, we can directly prove closure property under pullback along fibrations for strong homotopy equivalences.

Lemma 5.27. *Let $\Gamma \in \widehat{\mathbf{\Delta}}$, $A \in \mathsf{Ty}(\Gamma)$ a fibrant type, and $f\colon \Theta \to \Gamma$ a strong homotopy equivalence. Then $f^+\colon \Theta.Af \to \Gamma.A$ is a strong homotopy equivalence.*

Proof. Let $g\colon \Gamma \to \Theta$ be the homotopy inverse of f, and h, k the corresponding homotopies. Consider the lifting problem

$$\begin{tikzcd}
\Gamma.A \arrow[rr, "="] \arrow[d] & & \Gamma.A \arrow[d] \\
\Gamma.A \times \mathbf{\Delta}[1] \arrow[r] & \Gamma \times \mathbf{\Delta}[1] \arrow[r, "k"'] & \Gamma.
\end{tikzcd}$$

The left vertical map is a generalised open prism inclusion induced by $0 \to \Gamma.A$, while the right vertical map is a Kan fibration by assumption. Therefore, there exists a diagonal lift

$$k'\colon \Gamma.A \times \mathbf{\Delta}[1] \to \Gamma.A.$$

Now consider the diagram

$$\begin{tikzcd}
\Gamma.A \amalg_{\Theta.Af} (\Theta.Af \times \mathbf{\Delta}[1]) \arrow[r] \arrow[d] & \Gamma.A \times \mathbf{\Delta}[1] \arrow[r, "k'"] & \Gamma.A \arrow[d] \\
\Gamma \amalg_{\Theta} (\Theta \times \mathbf{\Delta}[1]) \arrow[r, "{[g,h]}"'] & \Theta \arrow[r, "f"'] & \Gamma.
\end{tikzcd}$$

The definition of k', together with the fact that g is a homotopy inverse for f, imply that this diagram commutes, and therefore the universal property of the pullback of lemma 3.2 determines a map

$$\Gamma.A \amalg_{\Theta.Af} (\Theta.Af \times \mathbf{\Delta}[1]) \to \Theta.Af.$$

In particular, we get a map $g' \colon \Gamma.A \to \Theta.Af$, as well as a homotopy

$$h' \colon \Theta.Af \times \mathbf{\Delta}[1] \to \Theta.Af.$$

One can now easily verify that g' constitutes a strong homotopy inverse of f^+ via the homotopies h' and k', and therefore f^+ is a strong homotopy equivalence. □

To effectively use the notion of strong homotopy equivalence and the closure property expressed by lemma 5.27, we need some basic examples to start with.

Lemma 5.28. *Generalised open prism inclusions are strong homotopy equivalences.*

Proof. Fix an arbitrary monomorphism $A \to B$, and let

$$i \colon B \amalg_A (A \times \mathbf{\Delta}[1]) \to B \times \mathbf{\Delta}[1]$$

be the corresponding generalised open prism inclusion. Denote by ι_1 and ι_2 the two canonical inclusions into the pushout $B \amalg_A (A \times \mathbf{\Delta}[1])$.

Define a homotopy inverse g as the composition

$$g \colon B \times \mathbf{\Delta}[1] \to B \to B \amalg_A (A \times \mathbf{\Delta}[1])$$

of the projection onto the first component, followed by the canonical inclusion of the base of the prism. On elements, g is defined by $g(b, t) = \iota_1(b)$.

Define homotopies $h \colon gi \sim \mathsf{id}$ and $k \colon ig \sim \mathsf{id}$ as follows:

$$\begin{aligned} h(\iota_1(b), s) &= \iota_1(b), \\ h(\iota_2(a, t), s) &= \iota_2(a, t \wedge s), \\ k(b, t, s) &= (b, t \wedge s). \end{aligned}$$

A simple verification shows that h and k are well defined homotopies as stated, and that the further condition of definition 5.19 is satisfied. □

We are now ready to prove the main result of this section.

Proof of proposition 5.26. It is enough to show that $\Gamma.\Pi(X, Y) \to \Gamma$ has the right lifting property with respect to all open prism inclusions. Without loss of generality, we can limit ourselves to lifting problems of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & B.\Pi(X, Y) \arrow[d] \\ B \arrow[r, "="'] & B, \end{tikzcd}$$

where i is an open prism inclusion. By the universal property of Π-types, it is enough to find a diagonal lift for the corresponding square

$$\begin{tikzcd} A.Xi \arrow[r] \arrow[d, "i^+"'] & B.X.Y \arrow[d] \\ B.X \arrow[r, "="'] & B.X. \end{tikzcd}$$

By lemma 5.28, i is a strong homotopy equivalence, and lemma 5.27 implies that the map $A.Xi \xrightarrow{i^+} B.X$ is a strong homotopy equivalence as well. Now, the map $p_Y : B.X.Y \to B.X$ is a fibration, since Y is a fibrant type. Therefore, i^+ has the left lifting property with respect to p_Y by lemma 5.20. $\square$

5.4 Contractibility and equivalences

By taking the type-theoretic definition of contractibility and translating it into simplicial sets using the type formers developed above, we get the following:

Definition 5.29. Let $\Gamma \in \widehat{\mathbf{\Delta}}$. A type $A \in \mathsf{Ty}(\Gamma)$ is said to be *contractible* if there is a term $c \in \mathsf{Tm}_\Gamma(A)$ (called its *centre of contraction*) and a commutative triangle:

$$\begin{tikzcd} \Gamma.A \arrow[rr] \arrow[dr, "c^+"'] & & \Gamma.A.A.\mathsf{Eq}_A \arrow[dl] \\ & \Gamma.A.A. & \end{tikzcd}$$

The second condition in definition 5.29 can be expressed by saying that there exists a homotopy over Γ between cp_A and the identity (as maps $\Gamma.A \to \Gamma.A$). In fact, we can also obtain a characterisation of contractible types purely in terms of lifting properties.

Proposition 5.30. *A type A over Γ is contractible if and only if it has the right lifting property with respect to monomorphisms.*

Proof. Let A be contractible, with centre of contraction c, and homotopy h between cp_A and id. Let $f : X \to Y$ be a monomorphism, and consider a lifting problem

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & \Gamma.A \arrow[d, "p_A"] \\ Y \arrow[r, "v"'] & \Gamma. \end{tikzcd}$$

Define a map $\ell' : Y \to \Gamma.A$ as $\ell' = cv$. Now, $p_A \ell' = v$, but ℓ' is not quite a lift, since the upper triangle does not necessarily commute. However, we can consider another lifting problem:

$$\begin{tikzcd}
Y \amalg_X (X \times \mathbf{\Delta}[1]) \arrow[r] \arrow[d, "i"'] & \Gamma.A \arrow[d] \\
Y \times \mathbf{\Delta}[1] \arrow[r] & \Gamma,
\end{tikzcd}$$

where i is the 0-oriented generalised open prism inclusion determined by f, and the top horizontal map is obtained from c and h. This problem has a lift m, because p is a fibration, and it is now easy to check that precomposing m with the map $\mathsf{id} \times (1) \colon Y \to Y \times \mathbf{\Delta}[1]$ gives a lift for the original problem.

Conversely, assume $p_A \colon \Gamma.A \to \Gamma$ has the right lifting property with respect to monomorphisms. In particular, the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \Gamma.A \arrow[d] \\
\Gamma \arrow[r, "\mathsf{id}"'] & \Gamma.
\end{tikzcd}$$

has a lift $c \colon \Gamma \to \Gamma.A$, and we have found a centre of contraction. The required homotopy can then be obtained as a lift of the diagram

$$\begin{tikzcd}
\Gamma.A \times \partial\mathbf{\Delta}[1] \arrow[r, "{[cp_A, \mathsf{id}]}"] \arrow[d] & \Gamma.A \arrow[d] \\
\Gamma.A \times \mathbf{\Delta}[1] \arrow[r] & \Gamma.
\end{tikzcd}$$

□

A map with the right lifting property with respect to all monomorphisms is called a *trivial fibration*. Therefore contractible types are those such that their display map is a trivial fibration.

Lemma 5.31. *A map $p \colon Y \to X$ of simplicial sets is a trivial fibration if and only if it has the right lifting property with respect to boundary inclusions.*

Proof. Since boundary inclusions are monomorphisms, one direction is obvious. Conversely, suppose p has the right lifting property with respect to boundary inclusions, and let $i \colon A \to B$ be an arbitrary monomorphism. By lemma 5.10 we can decompose i as a colimit of pushouts of coproducts of boundary inclusions. The conclusion then follows immediately from lemma 5.14. □

Note that a consequence of lemma 5.31 is that if the display map of a type A is a trivial fibration, then in particular A is fibrant.

We are now equipped to understand the type-theoretic notion of equivalence in terms of the simplicial model. Recall that a function in homotopy type theory is defined to be an equivalence if its fibres are contractible. A function $f\colon X \to Y$ between fibrant types over Γ corresponds to a morphism $\Gamma.X \to \Gamma.Y$ over Γ. The fibres of f form a fibrant type $F_f \in \mathsf{Ty}(\Gamma.Y)$. Translating the type-theoretic definition of fibre, we get that $F_f = \Sigma(X, E_f)$, where E_f is Eq_Y with its first argument being substituted by f.

We can fit all these types into a diagram

$$\begin{tikzcd}
\Gamma.Y.F_f \arrow[r, "\cong"] \arrow[ddr, "p_{F_f}"'] & \Gamma.X.Y.E_f \arrow[r] \arrow[d] & \Gamma.Y.Y.\mathsf{Eq}_Y \arrow[d] \\
& \Gamma.X.Y \arrow[r, "{(f,\mathsf{id})}"'] \arrow[d] & \Gamma.Y.Y \\
& \Gamma.Y, &
\end{tikzcd}$$

where the top right square is a pullback, and it follows immediately from proposition 5.30 that f is an equivalence if and only if p_{F_f} is a trivial fibration.

5.5 Fibrancy and univalence of the universe

For the purposes of this section, we will say that a simplicial map is a *strong equivalence* if it factors as a section of a trivial fibration followed by a trivial fibration. This allows us to determine whether a map is an equivalence before knowing that its domain is fibrant. In fact, we have the following:

Lemma 5.32. *Let $\Gamma \in \widehat{\mathbf{\Delta}}$, A, B types over Γ, with B fibrant. If $f\colon A \to B$ is a strong equivalence over Γ, then A is fibrant, and f is an equivalence.*

Proof. Factor f as $A \xrightarrow{i} E \xrightarrow{p} B$, where p is a trivial fibration and i has a retraction r which is a trivial fibration. Since B is fibrant, so is E. The map $\Gamma.A \to \Gamma$ is a retract of $\Gamma.E \to \Gamma$, hence it is a fibration.

Now every type is fibrant, and a trivial fibration between fibrant types is an equivalence. Therefore r and p are equivalences, and so is i. It follows that $f = pi$ is also an equivalence. ☐

Lemma 5.33. *Let $\Gamma \in \widehat{\mathbf{\Delta}}$,*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d, "f"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}$$

a pullback square of types over Γ, with B fibrant, and define A' as the pullback

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & \Pi(\mathbf{\Delta}[1], D) \arrow[d] \\ B \times C \arrow[r] & D \times D. \end{tikzcd}$$

Assume that the one of the two canonical maps

$$\Pi(\mathbf{\Delta}[1], B) \to \Pi(\mathbf{\Delta}[1], D) \times_D B \tag{3}$$

is a trivial fibration.

Then the canonical map $A \to A'$ is a strong equivalence.

Proof. We can form a diagram as follows:

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \Pi(\mathbf{\Delta}[1], B) \arrow[d] & \\ A' \arrow[r] \arrow[dd] & \Pi(\mathbf{\Delta}[1], B) \times_D B \arrow[r] \arrow[d] & B \arrow[d] \\ & \Pi(\mathbf{\Delta}[1], D) \arrow[r] \arrow[d] & D \\ C \arrow[r] & D, & \end{tikzcd}$$

where the top and right squares are pullbacks by definition. It is not hard to check that the bottom left square is also a pullback. If we now form the pullback squares

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \Pi(\mathbf{\Delta}[1], B) \arrow[d] \\ A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D, \end{tikzcd}$$

the map $E \to A$ is a trivial fibration. The map $A \to A'$ factors through E, and the first factor $A \to E$ is a section of the above trivial fibration. The second factor $E \to A'$ is a pullback of the map (3), hence a trivial fibration. Thus, we have proved that the map $A \to A'$ is a strong equivalence. ☐

Proposition 5.34 (Equivalence extension property)**.** *Let $i\colon \Theta \to \Gamma$ be a monomorphism, $X_0 \in \mathsf{Ty}(\Theta)$, $Y_1 \in \mathsf{Ty}(\Gamma)$, and $\alpha\colon \Theta.X_0 \to \Theta.Y_1 i$ an equivalence over Θ. Then* *there exists a type $Y_0 \in \mathsf{Ty}(\Gamma)$, and an equivalence $\beta\colon Y_0 \to Y_1$, such that the pullback of β along i is isomorphic to α.*

Proof. The following proof is adapted from [10].

Set $X_1 = Y_1 i$. To define an extension of $\alpha\colon X_0 \to X_1$ to the context Γ, we will make use of the presheaf cwf structure on $\widehat{\mathbf{\Delta}}$. Since i is a monomorphism, it has $\mathcal{V}$-small fibres, and therefore there exists a (not necessarily fibrant) type $P \in \mathsf{Ty}'(\Gamma)$ such that $\Theta \cong \Gamma.P$. For $j = 0, 1$, let $\widetilde{X}_j = \Pi(P, X_j)$, and note that α induces an equivalence $\widetilde{X}_0 \to \widetilde{X}_1$.

Since $X_1 = Y_1 i$, there is a map $Y_1 \to \widetilde{X}_1$ over Γ corresponding to the identity $X_1 \to X_1$ via the universal property of Π-types.

Therefore, we have a map $\widetilde{X}_0.Y_1 \to \widetilde{X}_1.\widetilde{X}_1$. Taking pullbacks, we obtain types Y_0 and Y_0' as follows:

$$\begin{tikzcd}
Y_0 \arrow[r] \arrow[d] & \widetilde{X}_1 \arrow[d] \\
Y_0' \arrow[r] \arrow[d] & \Pi(\mathbf{\Delta}[1], \widetilde{X}_1) \arrow[d] \\
\widetilde{X}_0 \times Y_1 \arrow[r] & \widetilde{X}_1 \times \widetilde{X}_1.
\end{tikzcd}$$

Let $\beta\colon Y_0 \to Y_1$ be the composition of the canonical map $Y_0 \to \widetilde{X}_0.Y_1$ with the projection to Y_1. It follows easily from lemma 5.35 below that the pullback of β along i is isomorphic to α. It is then enough to show that β is a strong equivalence.

First note that factoring the map $\widetilde{X}_0.Y_1 \to \widetilde{X}_1.\widetilde{X}_1$ through $\widetilde{X}_0.\widetilde{X}_1$, one obtains pullback squares

$$\begin{tikzcd}
Y_0' \arrow[r] \arrow[d] & \widetilde{E} \arrow[r] \arrow[d] & \Pi(\mathbf{\Delta}[1], \widetilde{X}_1) \arrow[d] \\
\widetilde{X}_0 \times Y_1 \arrow[r] \arrow[d] & \widetilde{X}_0 \times \widetilde{X}_1 \arrow[r] \arrow[d] & \widetilde{X}_1 \times \widetilde{X}_1 \\
Y_1 \arrow[r] & \widetilde{X}_1, &
\end{tikzcd}$$

where $\widetilde{E} = \Pi(P, E)$, and E is defined by the following pullback over Θ:

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \Pi(\mathbf{\Delta}[1], X_1) \arrow[d] \\ X_0 \times X_1 \arrow[r, "(\alpha,\mathsf{id})"'] & X_1 \times X_1. \end{tikzcd}$$

Since α is an equivalence, the map $E \to X_0 \times X_1 \to X_1$ is a trivial fibration. Lemma 5.36 then implies that $\widetilde{E} \to \widetilde{X}_1$ is a trivial fibration. It follows that the map $Y_0' \to Y_1$ is a trivial fibration as well.

Now we want to apply lemma 5.33 to the pullback square

$$\begin{tikzcd} Y_0 \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ \widetilde{X}_0 \arrow[r] & \widetilde{X}_1, \end{tikzcd}$$

so we need to check that the induced map $\Pi(\mathbf{\Delta}[1], Y_1) \to \Pi(\mathbf{\Delta}[1], \widetilde{X}_1) \times_{\widetilde{X}_1} Y_1$ is a trivial fibration. This follows from an argument similar to the proof of proposition 5.26, and the fact that $\Pi(\mathbf{\Delta}[1], Y_1) \to Y_1$ is a fibration, since Y_1 is fibrant.

Therefore, lemma 5.33 implies that the map $Y_0 \to Y_0'$ is a strong equivalence. Now β factors as a strong equivalence followed by a trivial fibration, hence it is a strong equivalence, as claimed. □

Lemma 5.35. *Let $\mathcal{I}$ be a small category, $\Gamma \in \widehat{\mathcal{I}}$, and $P \in \mathsf{Ty}(\Gamma)$ such that $\Gamma.P \to \Gamma$ is a monomorphism. Then for all types $X \in \mathsf{Ty}(\Gamma.P)$, the canonical map $\epsilon\colon \Gamma.P.\Pi(P, X) \to \Gamma.P.X$ is an isomorphism.*

Proof. Note that P is isomorphic to the unit type over the context $\Gamma.P$. Therefore, $\Pi(P, X) \cong \Pi(1, X) \cong X$ over $\Gamma.P$, and stability of Π implies that the isomorphism is given by ϵ. □

Lemma 5.36. *Let $\Gamma \in \widehat{\mathbf{\Delta}}$, and $A \in \mathsf{Ty}'(\Gamma)$ a (not necessarily fibrant) type. Then for any contractible type $B \in \mathsf{Ty}(\Gamma)$, the type $\Pi(A, B)$ is contractible.*

Proof. By an argument similar to the one in the proof of proposition 5.26, it is enough to show that for all monomomrphisms $i\colon \Theta \to \Gamma$, the map $i^+\colon \Theta.Ai \to \Gamma.A$ has the left lifting property with respect to $\Gamma.A.B \to \Gamma.A$. But i^+ is the pullback of i along $p_A\colon \Gamma.A \to \Gamma$, hence it is a monomorphism, and therefore it has the left lifting property with respect to trivial fibrations. □

In order to express proposition 5.34 as a lifting property, we now introduce a type that classifies equivalences. In the context $\mathcal{U} \times \mathcal{U}$, let X, Y be the types determines by the two canonical projections $\mathcal{U} \times \mathcal{U} \to \mathcal{U}$, and denote by $X \approx Y$ the type of equivalences between X and Y. Let Equiv be the corresponding extended context. Composing the display map with projections gives canonical maps $\pi_i \colon \mathsf{Equiv} \to \mathcal{U}$ for $i = 0, 1$. Note that, by construction, a morphism $\Gamma \to \mathsf{Equiv}$ corresponds to a choice of two types over Γ and an equivalence between them.

Proposition 5.37. *The map* $\mathsf{Equiv} \xrightarrow{\pi_1} \mathcal{U}$ *is a trivial fibration.*

Proof. A lifting problem for π_1 against a monomorphism $\Theta \to \Gamma$ corresponds exactly to the data of proposition 5.34. The existence of the type Y_0 and the equivalence β would imply the existence of a solution for this lifting problem, except the top triangle only commutes up to isomorphism. This can be resolved by regarding Y_0 as a presheaf over $\mathbf{\Delta}/\Gamma$, and defining a new presheaf Y_0' that coincides with X_0 *on the nose* on the subcategory $\mathbf{\Delta}/\Theta$, and with Y_0 outside. It is clear that Y_0' is isomorphic to Y_0, hence β can be transported to an equivalence between Y_0' and Y_1, giving an actual lift. $\square$

The choice of π_1 was important in the proof of proposition 5.34, but once we get this result for one of the π_i, the other easily follows.

Lemma 5.38. *The map* $\mathsf{Equiv} \xrightarrow{\pi_0} \mathcal{U}$ *is a trivial fibration.*

Proof. In the context $\mathcal{U} \times \mathcal{U}$, there is an obvious equivalence between $X_0 \approx X_1$ and $X_1 \approx X_0$. Therefore, there is an equivalence $\phi \colon \mathsf{Equiv} \to \mathsf{Equiv}$ such that

$$\begin{tikzcd} \mathsf{Equiv} \arrow[rr, "\phi"] \arrow[dr, "\pi_0"'] & & \mathsf{Equiv} \arrow[dl, "\pi_1"] \\ & \mathcal{U} & \end{tikzcd}$$

commutes. It follows that π_0 is an equivalence. Since it is also a fibration, it is a trivial fibration. $\square$

Corollary 5.39. *The simplicial set* $\mathcal{U}$ *is a Kan complex.*

Proof. Since the reflexivity map $\mathcal{U} \to \Pi(\mathbf{\Delta}[1], \mathcal{U})$ has the left lifting property with respect to fibrations (the proof of proposition 5.24 applies in this case, even though we do not yet know that $\mathcal{U}$ is fibrant), we get a lift $c \colon \Pi(\mathbf{\Delta}[1], \mathcal{U}) \to \mathsf{Equiv}$ in the diagram

$$\begin{tikzcd} \mathcal{U} \arrow[r, "i"] \arrow[d] & \mathsf{Equiv} \arrow[d] \\ \Pi(\mathbf{\Delta}[1], \mathcal{U}) \arrow[r] & \mathcal{U} \times \mathcal{U}, \end{tikzcd}$$

where i is the map that sends a type X to the identity equivalence on X.

Now consider a lifting problem

$$\begin{tikzcd}
\mathbf{\Lambda}^k[n] \arrow[r, "u"] \arrow[d] & \mathcal{U} \arrow[d] \\
\mathbf{\Delta}[n] \arrow[r, "v"'] & 1.
\end{tikzcd}$$

Recall from the proof of proposition 5.16 that horn inclusions are retracts of generalised open prism inclusions, so we have a diagram

$$\begin{tikzcd}
\mathbf{\Lambda}^k[n] \arrow[r, "f"] \arrow[d] & \mathbf{\Delta}[n] \amalg_{\mathbf{\Lambda}^k[n]} (\mathbf{\Lambda}^k[n] \times \mathbf{\Delta}[1]) \arrow[r, "g"] \arrow[d] & \mathbf{\Lambda}^k[n] \arrow[d] \\
\mathbf{\Delta}[n] \arrow[r, "f"'] & \mathbf{\Delta}[n] \times \mathbf{\Delta}[1] \arrow[r, "g"'] & \mathbf{\Delta}[n],
\end{tikzcd}$$

where the top and bottom horizontal arrows compose to identities. We will assume that the generalised open prism inclusion is 0-oriented, the other case being completely analogous. Let $w_1 \colon \mathbf{\Lambda}^k[n] \to \Pi(\mathbf{\Delta}[1], \mathcal{U})$ and $w_2 \colon \mathbf{\Delta}[n] \to \mathcal{U}$ the maps obtained from the two components of ug via the universal property of Π types. We can reassemble this data into a new lifting problem:

$$\begin{tikzcd}
\mathbf{\Lambda}^k[n] \arrow[r, "w_1"] \arrow[d] & \Pi(\mathbf{\Delta}[1], \mathcal{U}) \arrow[r, "c"] & \mathsf{Equiv} \arrow[d, "\pi_0"] \\
\mathbf{\Delta}[n] \arrow[rr, "w_2"'] & & \mathcal{U},
\end{tikzcd}$$

which we know has a solution $\ell \colon \mathbf{\Delta}[n] \to \mathsf{Equiv}$, because π_1 is a trivial fibration (hence in particular a fibration). One can now easily check that $\pi_0 \ell f$ is a lift for the original problem. □

Remark 5.40. Assuming the existence of a further Grothendieck universe $\mathcal{V}'$ contained in $\mathcal{V}$, we can repeat the construction of $\mathcal{U}$ using $\mathcal{V}'$ instead of $\mathcal{V}$, obtaining a smaller universe $\mathcal{U}'$ for the simplicial model. Since now $\mathcal{U}'$ is a $\mathcal{V}$-small presheaf, and a fibrant type by corollary 5.39, it satisfies the definition of a small universe, which means that it can be used to interpret a type theory with a universe type former.

We conclude our construction with a result summarising what we have achieved.

Theorem 5.41. *The cwf structure of fibrant types on simplicial sets admits Σ, Π and intensional equality type structures, as well as a small univalent universe.*

Proof. We constructed Σ-types in section 5.3.2, Π-types in section 5.3.5, intensional equality types in section 5.3.4. Corollary 5.39 proves that the universe

$\mathcal{U}$ constructed in section 5.2 is fibrant, and hence can be thought of as a fibrant type (modulo size issues as explained in remark 5.40), and hence as a small universe.

It remains to observe that proposition 5.37 translates to one of the equivalent type-theoretic formulations of univalence. Therefore $\mathcal{U}$ is univalent, completing the construction. $\square$

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